OPERATORS SIMILAR TO THEIR ADJOINTS

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This note is motivated by the following theorem of I. H. Sheth [4].

THEOREM. *Let T be a hyponormal operator ($TT^* \leqq T^*T$) and suppose that $S^{-1}TS = T^*$ where $0 \notin W(S)^-$. Then T is selfadjoint.*

Here, and in what follows, all operators will be bounded linear transformations from a fixed Hilbert space into itself. $W(S)^-$ is the closure of the *numerical range* $W(S) = \{\langle Sx, x\rangle : \|x\| = 1\}$ of S. We will make use of the fact that $W(S)^-$ is convex and contains the spectrum $\sigma(S)$ of S.

The purpose of this note is to present a simple theorem which explains the above result. Before doing this, however, a few remarks are pertinent.

To begin with, the technique of [4] actually proves a much better result:

THEOREM 1. *If T is any operator such that $S^{-1}TS = T^*$, where $0 \notin W(S)^-$, then the spectrum of T is real.*

PROOF. It is clearly enough to show that the boundary of $\sigma(T)$ lies on the real axis. Since this is a subset of the approximate point spectrum of T, it suffices to show that if (x_n) is a sequence of unit vectors such that $(T-\lambda)x_n \to 0$, then λ is real. This latter assertion follows from the inequality

$$\begin{aligned} |(\bar{\lambda} - \lambda)\langle S^{-1}x_n, x_n\rangle| &= |\langle (T^* - \lambda)S^{-1}x_n, x_n\rangle - \langle (T^* - \bar{\lambda})S^{-1}x_n, x_n\rangle| \\ &\leqq \|(T^* - \lambda)S^{-1}x_n\| + \|S^{-1}\| \, \|(T - \lambda)x_n\| \\ &= \|S^{-1}(T - \lambda)x_n\| + \|S^{-1}\| \, \|(T - \lambda)x_n\| \\ &\leqq 2\|S^{-1}\| \, \|(T - \lambda)x_n\| \end{aligned}$$

and the fact that $0 \notin W(S)^-$ implies $0 \notin W(S^{-1})^-$.

To recover Sheth's Theorem from Theorem 1 we need only observe that if N is hyponormal, then $W(N)^-$ is the convex hull of $\sigma(N)$.

It is worth noting that Theorem 1 includes a result of Beck and Putnam. To state this result, recall that a unitary operator U is *cramped* [2] if $\sigma(U)$ is contained in an arc of the unit circle with central angle less than Π.

COROLLARY [1], [2]. *If N is a normal operator which is unitarily*

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equivalent to its adjoint by a cramped unitary operator U, then N is selfadjoint.

PROOF. Since U is normal, $W(U)^-$ is the convex hull of the spectrum of U and so the hypothesis insures that $0 \notin W(U)^-$.

Theorem 1 fails if the operator S is merely required to be invertible. Even normality of both S and T does not help, as the following example shows:

EXAMPLE 1. Let T be the bilateral shift ($Te_n = e_{n+1}$) on the span l^2 of the orthonormal set $\{e_n\}_{-\infty}^{+\infty}$ and let S be the selfadjoint unitary defined by $Se_n = e_{-n}$. Then $S^{-1}TS = T^{-1} = T^*$, but the spectrum of T is not real.

Here is the promised generalization:

THEOREM 2. *If $S^{-1}TS = T^*$ where $0 \notin W(S)^-$, then T is similar to a selfadjoint operator.*

PROOF. Since $W(S)^-$ is convex and does not contain 0, we can separate 0 from $W(S)^-$ by a half-plane. If necessary, we may replace S by $e^{i\theta}S$ for suitably chosen θ to insure that this half-plane is $\mathrm{Re}\, z \geqq \epsilon$ for some $\epsilon > 0$. Then, if $A = \frac{1}{2}(S + S^*)$, the numerical range of A ($= \mathrm{Re}\, W(S)$) lies on the real axis to the right of ϵ, hence A is positive and invertible. Since $TA = AT^*$, it follows that $A^{-1/2}TA^{1/2}$ is selfadjoint.

The proof of Theorem 2 shows that if $TS = ST^*$ where S is positive and invertible, then T is similar to a selfadjoint operator. Both assumptions on S are essential here. Thus, Example 1 shows that the positivity condition on S cannot be omitted, and Dieudonné [3] has given an example which shows that the conditions $TS = ST^*$, $S > 0$ do not imply that the spectrum of T is real.

The conclusion of Theorem 2 cannot be strengthened either:

EXAMPLE 2. The conditions $S^{-1}TS = T^*$, $0 \notin W(S)^-$ do not imply that T is normal.

Here it suffices to take $T = SB$ where S is positive, B is selfadjoint, and S and B do not commute.

The converse of Theorem 2 is also valid. That is, if T is similar to a selfadjoint operator, then T is similar to T^* and the similarity can be implemented by an S with $0 \notin W(S)^-$.

PROOF. If $R^{-1}TR$ is selfadjoint, then $(RR^*)^{-1}T(RR^*) = T^*$, and $0 \notin W(RR^*)^-$ because RR^* is positive and invertible.

Thus Theorem 2 locates the scalar operators with real spectra in the larger class $\mathcal{C}$ of operators which are similar to their adjoints.

In conclusion I would like to thank P. A. Fillmore for a stimulating

discussion of Sheth's Theorem. He supplied Example 1. (The referee has observed that the same example occurs in the review of Sheth's paper [MR **33** #4685].)

NOTE ADDED IN PROOF. A paper of C. A. McCarthy [J. London Math. Soc. **39** (1964), 288–290] contains another generalization of the theorem of Beck and Putnam.

REFERENCES

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4. I. H. Sheth, *On hyponormal operators*, Proc. Amer. Math. Soc. **17** (1966), 998–1001.

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