APPROXIMATING SEMIGROUPS AND THE CONSISTENCY OF DIFFERENCE SCHEMES

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What hypotheses on the generators L_k ensure that the corresponding semigroups e^{tL_k} converge to a given semigroup e^{tL}? Assuming these are C_0 semigroups on a common Banach space $\mathcal{B}$, several alternative hypotheses have been proposed. We want to compare, and in fact to order, the three most important alternatives. The possibility of such an ordering is a question which was asked by Trotter [1], and appears to have remained unanswered.

Convergence is given the standard interpretation: For every $f \in \mathcal{B}$ and $T > 0$,

$$\sup_{0 \leqq t \leqq T} \| e^{tL_k} f - e^{tL} f \| \to 0 \qquad \text{as } k \to 0. \tag{1}$$

When the approximating semigroups are discrete—so that e^{tL_k} is replaced by the power S_k^n—the supremum is taken over positive multiples $t = nk \leqq T$. In this case the generator is the bounded operator $L_k = (S_k - I)/k$; the generators in the continuous case may of course be unbounded.

The strong convergence of a sequence of bounded operators, $A_n \to A$, is governed by familiar necessary and sufficient conditions: the A_n must be uniformly bounded, i.e. $\|A_n\| \leqq C$, and the convergence of $A_n f$ to Af must hold on a dense subset. Lax [2], [3] was the first to place the convergence of difference schemes into such a framework, and therefore the first to prove explicitly that the convergence of semigroups requires uniform boundedness, otherwise known as *stability*: (1) implies that

$$\| e^{tL_k} \| \leqq C(T) \quad \text{or} \quad \| S_k^n \| \leqq C(T) \tag{2}$$

as $k \to 0$, for $0 \leqq t \leqq T$. In the applications, it is this condition which has to be verified, and stability analysis has an enormous literature. In fact, the apparently rather mundane task of checking the usefulness of difference schemes requires exactly the techniques for a priori estimates which are fundamental to the theory of partial differential equations.

This note is concerned instead with the *second* requirement for

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strong convergence, namely convergence on a dense subset. This hypothesis is unsatisfactory as it stands, since it involves the exponentials $\exp[tL_k]$ or the powers S_k^n; we prefer a condition on their generators. It would be still better not to presuppose that L itself is a C_0 generator, but the hypotheses which we want to compare include this assumption.

In the discrete case, Lax replaced convergence on a dense subset by the following hypothesis of *consistency*: For each $T>0$, there is a set $\mathcal{A}$ dense in $D(L)$, and therefore dense in $\mathcal{B}$, such that for $f\in\mathcal{A}$

$$(\mathcal{L}) \qquad \sup_{0\leqq t\leqq T} \|(L_k - L)e^{tL}f\| \to 0 \qquad \text{as } k\to 0.$$

This extends directly to the continuous case, with the implicit requirement on $\mathcal{A}$ that $e^{tL}f\in D(L_k)$ for small k.

This hypothesis ($\mathcal{L}$), combined with stability, leads immediately to the classical proof of convergence for difference schemes. One solves the "error equation" with the identity

$$S_k^n - e^{nkL} = \sum_0^{n-1} S_k^{n-j-1}(S_k - e^{kL})e^{jkL} = k\sum S_k^{n-j-1}\left(L_k - \frac{e^{kL}-I}{k}\right)e^{jkL}.$$

For $f\in D(L)$,

$$\left\|\left(L - \frac{e^{kL}-I}{k}\right)e^{jkL}f\right\| \leqq \|e^{jkL}\| \left\|\left(L - \frac{e^{kL}-I}{k}\right)f\right\|$$

which goes to zero uniformly for $jk\leqq T$. Therefore, if $f\in\mathcal{A}$,

$$\|(S_k^n - e^{nkL})f\| \leqq nkC \sup\left\|(L_k - L)e^{jkL}f + \left(L - \frac{e^{kL}-I}{k}\right)e^{jkL}f\right\|$$

approaches zero, and convergence holds on the dense set $\mathcal{A}$. Then stability produces convergence for all f. This proof can be made to work also in the continuous case, but the result comes more easily from the following comparisons.

Kato [**4**] calls the set $\mathcal{C}\subseteq D(L)$ a *core* of the closed operator L if the pairs (f, Lf), for $f\in\mathcal{C}$, are dense in the graph of L. In other words, if the domain of L is restricted to $\mathcal{C}$, and the resulting operator is closed, the result is L. For many partial differential operators C_0^∞ serves as a core; in fact L is often defined precisely in this way, as the closure of its restriction to C_0^∞. In our problems, L is assumed to be a generator, so that $L-\beta$ has a bounded inverse for large β; $\mathcal{C}$ is then a core [**4**, p. 173] if and only if $L-\beta$ maps it onto a dense set.

Trotter has shown that the following consistency hypothesis—combined with stability in both the discrete and continuous cases—is sufficient to ensure convergence:

(𝔗) $\qquad L_k f \to Lf \qquad$ for f in some core $\mathcal{C}$ of L.

Thus when C_0^∞ is a core, it is enough to verify consistency just on this set. Unless the solutions *stay* in C_0^∞, more is required by hypothesis ($\mathcal{L}$).

Our first result is a comparison of these two hypotheses in the general case.

THEOREM 1. ($\mathcal{L}$) *implies* (𝔗), *but not conversely.*

PROOF. Suppose f lies in Lax's set $\mathfrak{A}$, so that $L_k e^{tL} f \to L e^{tL} f$ uniformly on $0 \leqq t \leqq T$. Then if

$$g = \int_0^T e^{-\beta t} e^{tL} f \, dt,$$

it follows that $L_k g \to Lg$. We show for large β that these elements g constitute a core, so that (𝔗) holds.

An integration by parts gives

$$(L - \beta) g = \int_0^T e^{-\beta t} \frac{\partial}{\partial t} (e^{tL} f) - \int_0^T \beta e^{-\beta t} e^{tL} f$$
$$= e^{-\beta t} e^{tL} f \Big|_0^T = (e^{-\beta T} e^{TL} - I) f.$$

If β is large, this last operator is invertible as well as bounded. Therefore it maps the dense set $\mathfrak{A}$ onto a dense set. This means that the elements $(L-\beta)g$ are dense, and we have a core.

For a counterexample to the converse, set $L = d^2/dx^2$ with its natural domain on the space $\mathfrak{B}$ of 2π-periodic functions in the sup norm. Define a centered difference operator by

$$\Delta_k f(x) = \tfrac{1}{2}(f(x + k) - f(x - k)).$$

Then an obvious choice for L_k is $k^{-2}\Delta_k^2$; instead we take

$$L_k = k^{-2} \Delta_k^2 - w k^{-15/4} \Delta_k^4 w,$$

where the operator w is multiplication by

$$w(x) = 2 + |\sin(x/2)|^{13/4}.$$

Both w and $1/w$ have third derivatives in $\mathrm{Lip}_{1/4}$ but not in $\mathrm{Lip}_{3/4}$.

We show that the set $\mathcal{C}$ of functions g for which wg has four derivatives is a core. Since it is clear that $g=w^{-1}(wg)$ has two derivatives, and that $L_kg\to Lg$, this verifies $(\mathfrak{J})$.

Certainly the set $\mathfrak{F}$ of functions with four derivatives is a core. Therefore it is enough to find for each $f\in\mathfrak{F}$ a sequence $g_n\in\mathcal{C}$ such that $g_n\to f$, $g_n''\to f''$. But if w_n is a sequence in $\mathfrak{F}$ with $w_n\to w$, $w_n'\to w'$, $w_n''\to w''$, we may take $g_n=fw_n/w$.

To see that $(\mathfrak{L})$ fails, recall that the solutions $u=e^{tL}f$ are analytic if $t>0$. The presence of the rough function w forces L_ku to blow up as $k\to 0$, unless $f\in D(L)$ and u happens to vanish at $x=0$. But if $e^{tL}f$ vanishes at the origin for $0<t\leqq T$, so does f, and such functions are not dense in the sup norm. Thus the required set $\mathcal{A}$ cannot exist.

A few alterations will make this counterexample stable so that it represents a *convergent* approximation which satisfies $(\mathfrak{J})$ but not $(\mathfrak{L})$. Recall that the continuous case is stable whenever the discrete case is: If $\|S_k^n\|\leqq C$, then

$$\begin{aligned}\|\exp[tL_k]\| &= \|\exp[-t/k]\exp[tS_k/k]\|\\ &= \exp[-t/k]\left\|\sum (t/k)^nS_k^n/n!\right\| \leqq C.\end{aligned}$$

To stabilize the discrete case we change scale, defining a new generator

$$\Lambda_k = L_{k^{1/4}}.$$

This makes $\|\Lambda_k\|=O(k^{-15/16})$, so that at least the first powers $S_k=I+k\Lambda_k$ are uniformly bounded as $k\to 0$. I believe that also stability holds in the sup norm, but the known a priori estimates for variable-coefficient operators fail to include this example. Therefore we switch to the L_2 norm, Λ_k becoming Hermitian and nonpositive. Since $k\|\Lambda_k\|\to 0$, this means that $0\leqq S_k\leqq I$ for small k, and stability is trivial: $\|S_k^n\|\leqq 1$.

On L_2, it is less easy to verify that $(\mathfrak{L})$ fails. We shall show that $wk^{-15/4}\Delta_k^4wu$ does not approach zero in L_2 when u is analytic, $u(0)\neq 0$. Assume this result; then condition $(\mathfrak{L})$ holds for $f\in D(L)$ only if

$$u(0) = e^{tL}f(0) = 0 \qquad \text{for } 0\leqq t\leqq T.$$

For $f=\sum a_ne^{inx}$ this is the same as

$$\sum a_n\exp[-n^2t] = 0,$$

which implies $a_n+a_{-n}=0$, forcing f to be odd. Since the odd functions are not dense in L_2, $(\mathfrak{L})$ fails.

Now we prove the result which was assumed above. If u is analytic and w has three L_2 derivatives,

$$wk^{-15/4}\Delta_k^4 wu = wuk^{-15/4}\Delta_k^4 w + O(k^{1/4}),$$

and we want $W_k = k^{-15/4}\Delta_k^4 w$ to misbehave at $x=0$. A suitable choice is

$$w(x) = \gamma + \sum_1^\infty \frac{\cos nx}{n^{17/4}},$$

where the constant γ is large enough to make $w>0$. In the first place $\|W_k\|$ approaches a constant:

$$\|W_k\|^2 = \sum_1^\infty k^{-15/2}\sin^8 nkn^{-17/2} \to I = \int_0^\infty x^{-17/2}\sin^8 x dx.$$

Furthermore, $W_k \to 0$ on any interval $(\epsilon, 2\pi-\epsilon)$. For this we compute

$$w''' = \sum \frac{\sin nx}{n^{5/4}}$$

and use the fact that $n^{-5/4}$ is a convex sequence approaching zero. According to Zygmund [**5**, p. 228] this gives w''' a continuous derivative on $(\epsilon, 2\pi-\epsilon)$, so on this interval $k^{-4}\Delta_k^4 w \to w^{(iv)}$ and $W_k \to 0$. Thus W_k is concentrated at the origin, and

$$\|wuW_k\|^2 \to |w(0)u(0)|^2 I.$$

Therefore $L_k u \to Lu$ only if $u(0)=0$. The hypothesis $(\mathfrak{J})$ holds in L_2 just as in the sup norm, so we have a stable counterexample to $(\mathfrak{J}) \Rightarrow (\mathfrak{L})$.

Now we come to the third and strongest consistency hypothesis, due to Kato:

$(\mathfrak{K})$ For large β, $(L_k-\beta)^{-1}$ converges strongly to $(L-\beta)^{-1}$.

THEOREM 2. *Given stability*: $(\mathfrak{J})$ *implies* $(\mathfrak{K})$, *but not conversely.*

PROOF. Stability ensures that for large β the inverses in $(\mathfrak{K})$ exist and are uniformly bounded as $k\to 0$. Suppose $(\mathfrak{J})$ holds for some core $\mathfrak{C}$. Reproducing Kato's argument,

$$(L_k-\beta)^{-1}f - (L-\beta)^{-1}f = (L_k-\beta)^{-1}(L-L_k)(L-\beta)^{-1}f \to 0$$

if $g=(L-\beta)^{-1}f\in\mathfrak{C}$. Since $L-\beta$ maps $\mathfrak{C}$ onto a dense set, such elements f are dense. Then the strong convergence in $(\mathfrak{K})$ follows by uniform boundedness.

In fact, Kato has shown that $(\mathfrak{K})$, together with stability, is *necessary* as well as sufficient for convergence. This also proves, indirectly, that $(\mathfrak{K})$ follows from $(\mathfrak{J})$ in the presence of stability.

For a counterexample to the converse, we take $L=0$ on the same space $L_2(0, 2\pi)$. The generators L_k will have rank one,

$$L_k f = -(f, g_k)g_k/k, \qquad \|g_k\| = 1.$$

This means that

$$\|L_k f - Lf\| = |(f, g_k)|/k,$$

whereas

$$\|\exp[tL_k]f - e^{tL}f\| = (1 - e^{-t/k})|(f, g_k)| < |(f, g_k)|$$

and

$$\|(I + kL_k)^n f - e^{nkL}f\| = |(f, g_k)|.$$

Thus, if $(f, g_k)\to 0$ for every f, we have convergence as well as stability. It then follows, from the necessity of $(\mathcal{K})$ or from direct calculation, that $(\mathcal{K})$ holds.

At the same time, in order that $(\mathfrak{I})$ shall fail, we choose g_k so that for $f = \sum a_n e^{inx}$,

$$(f, g_k)/k \to 0 \quad \text{implies} \quad a_0 = 0.$$

For this we let g_k be the unit vector

$$g_k = N^{-1/2} + e^{iNx}(1 - 1/N)^{1/2}, \qquad N = 1/k.$$

Clearly we do have convergence:

$$(f, g_k) = a_0 N^{-1/2} + a_N(1 - 1/N)^{1/2} \to 0 \qquad \text{as } k \to 0.$$

If we also have $L_k f \to Lf = 0$, so that

$$(f, g_k)/k = N^{1/2}(a_0 + N^{1/2}a_N(1 - 1/N)^{1/2}) \to 0,$$

it follows that $N^{1/2}a_N \to -a_0$. But if $a_0 \neq 0$ this means that

$$|a_N|^2 > |a_0|^2/2N \qquad \text{for large } N \quad \text{and} \quad \sum |a_N|^2 = \infty,$$

contradicting $f \in L_2$. Thus $L_k f \to Lf$ only if $a_0 = 0$; since such f are not dense in L_2, hypothesis $(\mathfrak{I})$ fails to hold. Of course $(\mathfrak{L})$ must also fail, despite the convergence of this example.

When L and L_k are differential or difference operators on L_2 with constant coefficients, the three versions of consistency coincide. In fact one need not assume that the equation $\partial u/\partial t = Lu$ is correct, i.e. that L is a generator. For these problems the Lax equivalence theorem becomes more precise:

stability and consistency $\Leftrightarrow$ correctness and convergence.

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