FIXED-POINT THEOREMS FOR CERTAIN CLASSES OF NONEXPANSIVE MAPPINGS[1]

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1. **Introduction.** A mapping f of a metric space M into itself is called *nonexpansive* if $d(f(x), f(y)) \leqq d(x, y)$ for each $x, y \in M$. For each $x \in M$, let $O(f^n(x))$ denote the sequence of iterates of $f^n(x)$, that is,

$$O(f^n(x)) = \bigcup_{i=n}^{\infty} \{f^i(x)\}, \qquad n = 0, 1, 2, \cdots,$$

where it is understood that $f^0(x) = x$. Our main purpose here is to prove fixed-point theorems for nonexpansive mappings f for which the diameters of the sets $O(f^n(x))$ satisfy a condition introduced below, a condition which is suggested by a consideration of the Banach Contraction Principle. For such mappings f, compactness of M is seen to imply that every sequence of iterates $\{f^n(x)\}$ of x converges to a fixed-point of f (which is not necessarily unique) while if M is a weakly compact, closed, and convex subset of a Banach space, then the existence of a fixed-point for f is established. In the final section we show how the results of this paper lead in an indirect way to a generalization of Theorem 3 of [**1**].

2. **Limiting orbital diameters.** For a subset A of M, let $\delta(A) = \sup \{d(x, y): x, y \in A\}$ denote the diameter of A, and let $f: M \to M$.

In general the sequence $\delta(O(f^n(x)))$ is nonincreasing and has limit $r(x) \geqq 0$. We call the number $r(x)$ (which may be infinite) the *limiting orbital diameter of f at x*, and introduce the following definition:

DEFINITION. If f is a mapping of M into itself which has the property that for each $x \in M$ the limiting orbital diameter $r(x)$ of f at x is less than $\delta(O(x))$ when $\delta(O(x)) > 0$, then f is said to have *diminishing orbital diameters*.

It is easy to give examples of nonexpansive mappings which have diminishing orbital diameters. For let $f: M \to M$ be such that for each $x \in M$ we have an $\alpha(x)$, $0 \leqq \alpha(x) < 1$, and $d(f(x), f(y)) \leqq \alpha(x) d(x, y)$ for each $y \in M$. Thus, for $n > 1$, $d(f(x), f^n(x)) \leqq \alpha(x) d(x, f^{n-1}(x))$. This gives

$$\sup_n d(f(x), f^n(x)) = \delta(O(f(x))) \leqq \sup_n \alpha(x) d(x, f^{n-1}(x))$$
$$= \alpha(x)\delta(O(x)).$$

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Hence,

$$r(x) = \lim_{n\to\infty} \delta(O(f^n(x))) \leqq \delta(O(f(x)) \leqq \alpha(x)\delta(O(x)) < \delta(O(x)),$$

if $0<\delta(O(x))$. Thus f has diminishing orbital diameters.

For the type of mapping above, the existence of a fixed point yields the following. Let $f(x_0)=x_0$. Then $d(x_0, f(x))\leqq\alpha(x_0)d(x_0, x)$. Also, $d(x_0, f^n(x))=d(f(x_0), f^n(x))\leqq\alpha(x_0)d(x_0, f^{n-1}(x))$. Hence an induction argument shows that $d(x_0, f^n(x))\leqq(\alpha(x_0))^n d(x_0, x)$, for each $n\geqq 1$. Thus for any $x\in M$ we have $\lim_{n\to\infty} f^n(x)=x_0$.

THEOREM 1. *Let M be a metric space and let f be a nonexpansive mapping of M into itself which has diminishing orbital diameters. Suppose for some $x\in M$ a subsequence of the sequence $\{f^n(x)\}$ of iterates of x has limit z. Then $\{f^n(x)\}$ has limit z and z is a fixed point of f.*

PROOF. Suppose $\lim_{k\to\infty} f^{n_k}(x)=z$. Then by a theorem of Edelstein [5, Theorem 1′], z generates an isometric sequence. This means that for given positive integers m and n,

$$d(f^m(z), f^n(z)) = d(f^{m+k}(z), f^{n+k}(z)), \qquad k = 1, 2, \cdots.$$

Therefore if k is any positive integer,

$$\begin{aligned}\delta(O(f(z))) &= \sup_{n\geqq 1} d(f(z), f^n(z)) \\ &= \sup_{n\geqq 1} d(f^k(z), f^{n+k-1}(z)) \\ &= \delta(O(f^k(z))).\end{aligned}$$

This implies

$$\lim_{k\to\infty} \delta(O(f^k(z))) = r(z) = \delta(O(f(z))).$$

But $r(z)=r(f(z))$. Since $r(f(z))=\delta(O(f(z)))$, the assumption that f has diminishing orbital diameters enables us to conclude $\delta(O(f(z)))=0$ and thus $f(z)$ is a fixed point of f. Continuity of f implies $\lim_{k\to\infty} f^{n_k+1}(x)=f(z)$. Thus if $\epsilon>0$ there is an integer k such that $d(f^{n_k+1}(x), f(z))<\epsilon$. The fact that $f(z)$ is a fixed-point and f is nonexpansive implies $d(f^n(x), f(z))<\epsilon$ if $n\geqq n_k+1$. Thus $\lim_{n\to\infty} f^n(x)=f(z)$. But since a subsequence of $\{f^n(x)\}$ has limit z, $z=f(z)$ completing the proof.

COROLLARY 1. *If M is any compact metric space and if f is any nonexpansive mapping of M into itself which has diminishing orbital diameters, then for each $x\in M$ the limiting orbital diameter $r(x)$ of f at x is zero, and the sequence $\{f^n(x)\}$ of iterates of x converges to a fixed-point of f.*

3. **Weakly compact sets.** The concept of diminishing orbital diameters has significant implications in noncompact settings. In this section we obtain a result which implies that for closed convex subsets of a Banach space, weak compactness is sufficient to ensure the *existence* of a fixed-point for nonexpansive mappings with diminishing orbital diameters.

First we introduce some notation. Let X be a Banach space. For a subset A of X, $\mathrm{cl\,co}\,A$ will denote the closed convex hull of A. For $x \in X$ and ρ a positive number, $\mathfrak{U}(x;\rho)$ will denote the closed spherical ball centered at x with radius ρ: $\mathfrak{U}(x;\rho)=\{z\in X: \|x-z\|\leqq\rho\}$.

THEOREM 2. *Let K be a bounded closed convex subset of a Banach space X, and let M be a weakly compact subset of X. If f is a nonexpansive mapping of K into K such that*

(i) *for each $x\in K$, $\mathrm{cl\,co}(O(x))\cap M\neq\emptyset$, and*

(ii) *f has diminishing orbital diameters,*

then there is a point $x\in M$ such that $f(x)=x$.

PROOF. If $\{K_\alpha\}$ is a descending chain of closed convex (hence weakly closed) subsets of K, each of which intersects M, then the weak compactness of M implies $(\bigcap K_\alpha)\cap M\neq\emptyset$. Thus we may use Zorn's Lemma to obtain a subset K_1 of K which is minimal with respect to being closed, convex, invariant under f, and having points in common with M. Let $M_1=K_1\cap M$.

Let $x\in K_1$ and suppose $\delta(O(x))>0$. We show this assumption leads to contradiction. By (ii) there is an integer N such that

$$\delta(O(f^N(x))) = r < \delta(O(x)).$$

Let $U=\{z\in K_1: \|z-f^n(x)\|\leqq r$ for almost all $n\}$. Since $\delta(O(f^N(x)))\leqq r$, $O(f^N(x))\subseteq U$ and thus $U\neq\emptyset$. If $y\in U$ then for some integer N_1, $\|y-f^n(x)\|\leqq r$ if $n\geqq N_1$. Since f is nonexpansive, $\|f(y)-f^{n+1}(x)\|\leqq r$ if $n+1\geqq N_1+1$, and thus U is mapped into itself. Clearly U is convex since spherical balls of radius r centered at each two points u_1, u_2 of U contain some common set $O(f^n(x))$. Thus a ball of radius r centered at any point of the segment joining u_1 and u_2 will also contain $O(f^n(x))$. Therefore, the closure $\overline{U}$ of U is convex and mapped into itself by f; (i) implies $\overline{U}\cap M\neq\emptyset$, and the minimality of K_1 implies $\overline{U}=K_1$.

Let $p\in K_1$. Then since $p\in\overline{U}$, if $\epsilon>0$ there is a point $p'\in U$ such that $\|p-p'\|<\epsilon$. For some integer N_2, $\|p'-f^n(x)\|\leqq r$ if $n\geqq N_2$. Therefore $\|p-f^n(x)\|\leqq r+\epsilon$ if $n\geqq N_2$. Hence

$$\mathrm{cl\,co}\,O(f^n(x)) \subseteq \mathfrak{U}(p; r+\epsilon), \qquad n\geqq N_2.$$

By (i), $\mathrm{cl\,co}\,O(f^n(x))\cap M_1\neq\emptyset$, and since M_1 is weakly compact there is a point t such that

$$t \in \left(\bigcap_{n=1}^{\infty} \operatorname{cl} \operatorname{co} O(f^n(x)) \right) \cap M_1.$$

Then $t \in \mathfrak{U}(p; r+\epsilon)$ for each ϵ. Thus $t \in \mathfrak{U}(p; r)$. Since this is true for each $p \in K_1$, it follows that

$$t \in \bigcap_{p \in K_1} \mathfrak{U}(p; r).$$

Therefore the set

$$S = \{ z \in K_1 \colon K_1 \subseteq \mathfrak{U}(z; r) \}$$

contains t, so S is nonempty.

The remainder of our argument follows the argument given in [7].

It is easily seen that S is closed and convex. Suppose for some $z \in S$, $f(z) \notin S$. Let $x \in H = \mathfrak{U}(f(z); r) \cap K_1$. Then $\|f(x) - f(z)\| \leqq \|x - z\|$ and $\|x - z\| \leqq r$. Because $f(z) \notin S$ by assumption, there is a point $x \in K_1$, such that $\|x - f(z)\| > r$. Hence H is a *proper* subset of K_1. Since H is closed, convex, and $H \cap M$ is nonempty (because $f(H) \subseteq H$), we have contradicted the minimality of K_1.

Therefore $f(S) \subseteq S$. But

$$\delta(S) \leqq r = \delta(O(f^N(x))) < \delta(O(x)) \leqq \delta(K_1)$$

so S is a proper subset of K_1. Again the minimality of K_1 is contradicted. Therefore the original assumption that $\delta(O(x)) > 0$ is incorrect, and $\delta(O(x)) = 0$. This implies $f(x) = x$.

COROLLARY 2. *If K is a closed, convex, weakly compact subset of X and if f is a nonexpansive mapping of K into itself which has diminishing orbital diameters, then f has a fixed point in K.*

The above corollary is obtained by observing that since K is weakly compact, condition (i) of the theorem holds trivially upon letting $M = K$.

It is not known whether Theorem 2 (or Corollary 2) remains true without the assumption of diminishing orbital diameters of f. This question is essentially equivalent to a question raised in [7] (as to whether the condition of "normal structure" is necessary for the theorem of [7]) which remains open. Similar results for nonexpansive mappings, without orbital constraints, are given by the authors in [1].

4. **Normal structure.** A very slight modification of the proof of Theorem 2 yields a theorem which is a generalization of Theorem 3 of [1].

Let A be a bounded subset of the Banach space X. A point $a \in A$ is a *nondiametral* point of A if

$$\sup\{\|x - a\| : x \in A\} < \delta(A).$$

A bounded convex subset K of X is said to have *normal structure* (Brodskii and Milman [3]) if for each subset H of K which contains more than one point there is a point $x \in H$ which is a nondiametral point of H.

THEOREM 3. *Let K be a bounded closed convex subset of a Banach space X, and let M be a weakly compact subset of K. If f is a nonexpansive mapping of K into K such that for each $x \in K$*

(i) *cl $\operatorname{co}(O(x)) \cap M \neq \emptyset$, and*

(ii) *cl $\operatorname{co}(O(x))$ has normal structure,*

then there is a point $x \in M$ such that $f(x) = x$.

PROOF. Define K_1 as in the proof of Theorem 2 and obtain the set U as follows: Suppose $\delta(K_1) > 0$. Let $x \in K_1$. By (ii) there is a point $y \in \operatorname{cl} \operatorname{co}(O(x))$ such that

$$\sup\{\|y - w\| : w \in \operatorname{cl} \operatorname{co}(O(x))\} = r < \delta(\operatorname{cl} \operatorname{co}(O(x))).$$

Let

$$U = \{z \in K_1 : O(f^n(x)) \subseteq \mathfrak{U}(x; r) \text{ for some } n\}.$$

Then $y \in U$ so U is not empty. The closure $\overline{U}$ of U is convex and mapped into itself by f. Therefore $\overline{U} = K_1$. Following the argument of Theorem 2, one sees that the set

$$S = \{z \in K_1 : K_1 \subseteq \mathfrak{U}(z; r)\}$$

is closed, convex, nonempty, and mapped into itself by f. But

$$\delta(S) \leqq r < \delta(O(x)) \leqq \delta(K_1),$$

so S is a proper subset of K_1 contradicting the minimality of K_1. Therefore $\delta(K_1) = 0$ and K_1 consists of a single point which is fixed under f.

Since compact convex sets have normal structure (this is essentially Lemma 1 of [4]), we have the following corollary.

COROLLARY 3. *If K is a closed convex weakly compact subset of X and if f is a nonexpansive mapping of K into K for which $O(x)$ is precompact for each $x \in K$, then f has a fixed-point in K.*

Precompactness of $O(x)$ does not in general imply f has diminishing orbital diameters. In fact, as a consequence of the above corollary, one might note that if f is a periodic isometry of K into K, f has a fixed-point.

Some examples of spaces which possess normal structure are given in [2].

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