A THEOREM ON THE HUREWICZ FIBERINGS

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1. **Introduction.** Let $p: E \rightarrow B$ be a Hurewicz fiber map; i.e., the map p has the path lifting property or, what is the same, the covering homotopy property holds for all topological spaces.

In [5], Stasheff showed that E has the homotopy type of a CW-complex if all fibers and B have the homotopy type of CW-complexes, and in [1], Allaud and Fadell showed that E is an ANR if all fibers and B are ANR's and E is a separable metric finite dimensional space. In this paper we prove the following theorem:

THEOREM. *Let $p: E \rightarrow B$ be a regular Hurewicz fiber map from a connected, locally compact separable metric (finite dimensional) space E onto an ANR base B. Assume that B is a generalized m-manifold over a principal ideal domain L without boundary, and that each fiber is an ANR generalized k-manifold over L without boundary. Then,* (1) *E is generalized $(m+k)$-manifold over L without asserting any local orientability. Moreover,* (2) *if some fiber is compact and orientable, then E is a locally orientable generalized manifold, and* (3) *if the cohomology dimension of $E \leqq 2$, or $=3$ and E is triangulable, then E is a topological manifold.*

By a generalized n-manifold (n-gm) we mean what Raymond and Wilder call a locally orientable generalized n-manifold or cohomology n-manifold (see [3] and [6]). If a Hurewicz fiber map has a lifting function which lifts a constant path to a constant path, then it is called regular. In [4], Raymond proved a converse to the theorem above.

If the fibering is locally trivial, it follows trivially by Theorem 6 of [3] that E is a gm, i.e., we have the following proposition:

PROPOSITION. *Let $p: E \rightarrow B$ be a locally trivial fiber map from E onto B. Assume that the fiber F and the base B are a k-gm and a m-gm over a principal ideal domain L with or without boundary, respectively. Then E is a $(k+m)$-gm over L with or without boundary.*

COROLLARY. *Let $p: E \rightarrow B$ be a Hurewicz fiber map from a connected locally connected compact (or p is a proper map) separable metric ANR space E onto a weakly locally contractible (wlc) and paracompact base B. Assume B is a m-gm over a principal ideal domain L with or without*

Received by the editors September 7, 1967.

boundary, and each fiber is a k-gm ($k \leqq 2$) over L with or without boundary and all fibers are homeomorphic. Then E is a ($k+m$)-gm over L with or without boundary.

The proof is immediate because the fibering is locally trivial by theorems of [2] and [4].

A part of the theorem is included in a portion of the author's dissertation prepared under Professor F. Raymond. I wish to express my thanks to Professor Raymond for his guidance and for discussions concerning this material. In particular, (2) of the theorem is largely due to him.

2. **Proof of the theorem.**

(1) We want to show that for each e in E there is an open set V in E containing e such that

$$\begin{aligned} H_r^s(V, V - e'; L) &\cong L \quad \text{if} \quad r = m + k \\ &\cong 0 \quad \text{if} \quad r \neq m + k \end{aligned} \qquad \text{for all } e' \text{ in } V.$$

(Here $H_r^s(X, A; L)$ is the rth relative singular homology group of the pair (X, A).) Since E is an ANR, E is lc_r^s for all r (locally connected up to dimension r in the homology sense). Hence the above implies that E is a ($m+k$)-gm (without asserting local orientability) by Proposition (3.4) of [4]. Since B is an ANR, B is uniformly contractible. Let U be a uniformly contractible open set containing $p(e)=b$. Then U is a m-gm over L. Since U is an ANR, U is a singular homology m-manifold (see [4] for the definition) by (3.4) of [4]. Since $F_b = p^{-1}(b)$ is also a singular homology manifold over L, $U \times F_b$ is a ($m+k$)-singular homology manifold over L. Therefore, we have

$$\begin{aligned} H_r^s(U \times F_b, U \times F_b - (b' \times e'); L) &= L \quad \text{if} \quad r = m + k \\ &= 0 \quad \text{if} \quad r \neq m + k \end{aligned}$$

for all $b' \in U$ and $e' \in F_b$.

Let H_b: $U \to B^I$ be such that, for each $b' \in U$, $H_b(b')(0) = b'$, $H_b(b')(1) = b$, and $H_b(b)(t) = b$ for all $t \in I$, where I is the unit interval. Then the map ϕ_b: $p^{-1}(U) \to U \times F_b$, defined by

$$\phi_b(e') = (p(e'), \lambda[e', H_b(p(e'))](1)),$$

is a fiber homotopy equivalence, and $\phi_b | F_b =$ identity on F_b, i.e., $\phi_b(e') = (b \times e')$ for each $e' \in F_b$, where λ is a regular lifting function for the fibering (E, B, p). Then

$$\phi_b: (p^{-1}(U), p^{-1}(U) - e') \to (U \times F_b, U \times F_b - (b \times e')),$$

for each $e' \in F_b$, is a fiber homotopy equivalence. Therefore

$$\begin{aligned} \overset{s}{H}_r(p^{-1}(U), p^{-1}(U) - e'; L) &\cong \overset{s}{H}_r(U \times F_b, U \times F_b - (b \times e'); L) \\ &= L \quad \text{if} \quad r = m + k \\ &= 0 \quad \text{if} \quad r \neq m + k \end{aligned} \qquad \text{for all } e' \in F_b.$$

But this is true for any point $b' \in U$ because U is uniformly contractible. Therefore, setting $p^{-1}(U) = V$, we have

$$\begin{aligned} \overset{s}{H}_r(V, V - e'; L) &= L \quad \text{if} \quad r = m + k \\ &= 0 \quad \text{if} \quad r \neq m + k \end{aligned} \qquad \text{for all } e' \in V.$$

This completes the proof of the first part of the theorem.

(2) Since B is an ANR gm, for each point $b \in B$, there is an open set U in B such that $U \ni b$ and is connected, orientable, and uniformly contractible. We may assume that F_b is connected because p can be factored as qp': $E \xrightarrow{p'} B' \xrightarrow{q} B$, where p' is a regular Hurewicz fiber map, B' is a 0-connected space, and q is a covering map such that $p'^{-1}(x)$ is a path component of $p^{-1}(q(x))$ for each $x \in B'$ by (2.10) of [**4**]. That is, we can replace the fibering by one with a connected fiber.

Let $\{W_\alpha\}$ be a covering of U such that each W_α is an open set with compact closure in U. Then $\{p^{-1}(W_\alpha)\}$ is an open covering of $p^{-1}(U)$. We note that $\mathrm{cl}(p^{-1}(W_\alpha)) = p^{-1}(\mathrm{cl}(W_\alpha))$ because p is an open map, where $\mathrm{cl}(W_\alpha)$ denotes the closure of W_α. We will show that

$$j_*\colon \overset{s}{H}_{m+k}(p^{-1}(U), p^{-1}(U - \mathrm{cl}(W_\alpha))) \to \overset{s}{H}_{m+k}(p^{-1}(U), p^{-1}(U) - (b' \times e'))$$

is bijective for each $b' \in W_\alpha$ and $e' \in F_{b'}$, where j_* is induced by the inclusion map

$$j\colon (p^{-1}(U), p^{-1}(U - \mathrm{cl}(W_\alpha))) \subset (P^{-1}(U), p^{-1}(U) - (b' \times e')).$$

Since $U \times F_{b'}$ is an orientable singular homology $(m+k)$-manifold and $\mathrm{cl}(W_\alpha) \times F_{b'}$ is connected and compact,

$$\begin{aligned} j_*\colon \overset{s}{H}_{m+k}((U, U - \mathrm{cl}(W_\alpha)) \times F_{b'}) &\to \overset{s}{H}_{m+k}((U, U - b') \times (F_{b'}, F_{b'} - e')) \\ &= \overset{s}{H}_{m+k}(U \times F_{b'}, U \times F_{b'} - (b' \times e')) \cong L, \qquad e' \in F_{b'}, \end{aligned}$$

is bijective.

Now consider the following commutative diagram

$$\begin{array}{ccc} \overset{s}{H}_{m+k}((U, U - \mathrm{cl}(W_\alpha)) \times F_{b'}) & \overset{\psi_{b'*}}{\cong} & \overset{s}{H}_{m+k}(p^{-1}(U), p^{-1}(U - \mathrm{cl}(W_\alpha))) \\ \downarrow \cong_{j_*} & & \downarrow j_* \\ \overset{s}{H}_{m+k}(U \times F_{b'}, U \times F_{b'} - (b' \times e')) & \overset{\psi_{b'*}}{\cong} & \overset{s}{H}_{m+k}(p^{-1}(U), p^{-1}(U) - (b' \times e')) \end{array}$$

where $\psi_{b'}$ is bijective given by the homotopy inverse of $\phi_{b'}$. Since $\psi_{b'_*} j_* \psi_{b'_*}^{-1}$ is bijective, the right hand j_* is bijective, i.e.,

$$j_*: \overset{s}{H}_{m+k}(p^{-1}(U), p^{-1}(U - \mathrm{cl}(W_\alpha))) \to \overset{s}{H}_{m+k}(p^{-1}(U), p^{-1}(U) - (b' \times e'))$$

is bijective. Now diagram holds for all $e' \in F_{b'}$, for fixed b'. But as U is uniformly contractible, a similar diagram holds for each $b \in W_\alpha$, thus the local orientability of $p^{-1}(U)$ is proved.

(3) Now in the case of the cohomology dimension of $E \leqq 2$, or $=3$ and E is triangulable, it is known that E is a topological manifold (see Chapter VII and IX of [6]). Q.E.D.

References

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