VALUATIONS ON A COMMUTATIVE RING

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By a valuation on a commutative ring $R$, we mean a pair $(v, \Gamma)$, where $\Gamma$ is an ordered (mult) group with a zero adjoined and $v$ is a map of $R$ onto $\Gamma$ satisfying

1. $v(xy) = v(x)v(y)$ for all $x, y \in R$,
2. $v(x+y) \leq \max\{v(x), v(y)\}$ for all $x, y \in R$.

The pair will usually be denoted simply by $v$.

The definition without the “onto” requirement is standard, but this requirement is crucial in obtaining our analogues of classical theorems on valuations on a field and results already used in [3]. Throughout this paper, “ring” means “commutative ring with 1” and subrings always contain that 1. Other conventions and notation used should be clear from context.

**Proposition 1.** Let $A$ be a subring of $R$, $P$ a prime ideal of $A$. Then the following are equivalent.

(i) For each subring $B$ of $R$, and ideal $Q$ of $B$ with $A \subseteq B$ and $Q \cap A = P$, one has $A = B$.

(ii) For $x \in R \setminus A$, there is an $x' \in P$ with $xx' \in A \setminus P$.

(iii) There is a valuation $v$ on $R$ with $A = \{x \in R \mid v(x) \leq v(1)\}$ and $P = \{x \in R \mid v(x) < v(1)\}$.

**Proof.** (i)$\Rightarrow$(ii) is straightforward using the “trick” used in [5] to show that a valuation on a subfield can be extended to an overfield.

(ii)$\Rightarrow$(iii). Write $x \sim y$ when $\{z \in R \mid xz \in P\} = \{z \in R \mid yz \in P\}$. One checks that $\sim$ is an equivalence relation on $R$. Let $v(x) = \{y \mid x \sim y\}$ and $\Gamma = \{v(x) \mid x \in R\}$. One checks: $v(x) < v(y)$ whenever $3z \in R$ with $zx \in P$ and $zy \in P$ and $v(x)v(y) = v(xy)$ are well defined and $(v, \Gamma)$ is a valuation on $R$ with the required properties.

(iii)$\Rightarrow$(i). Using standard techniques for fields, one shows $A = \{x \mid v(x) \leq v(1)\}$ is a subring of $R$ and $\{x \mid v(x) < v(1)\} = P$ is a prime ideal of $A$. If $x \notin A$ then $v(x)^{-1} = v(x')$ for some $x' \in P$, and since $xx' \notin A \setminus P$, the proposition follows.

We call pairs satisfying (i), (ii) and (iii) valuation pairs. Note they are the subject of exercise 7, Chapter 6 of [1].

**Proposition 2.** Two valuations $(v, \Gamma)$ and $(w, \Delta)$ determine the same
valuation pair if and only if there is an order isomorphism \( \phi : \Gamma \to \Lambda \) such that \( w = \phi \circ v \).

**Proof.** The same as for fields.

Henceforth we speak of the valuation determined by a valuation pair. Given a valuation \((v, \Gamma)\) on \(R\) we let

\[ A_v = \{ x \in R \mid v(x) \leq v(1) \} \quad \text{and} \quad P_v = \{ x \in R \mid v(x) < v(1) \}. \]

Let \((v, \Gamma)\) be a valuation on \(R\). A \(v\)-closed ideal \(\mathfrak{A}\) of \(A_v\) is an ideal such that \(x \in \mathfrak{A}, v(y) \leq v(x) \Rightarrow y \in \mathfrak{A}\).

**Proposition 3.** The \(v\)-closed ideals of \(A_v\) are linearly ordered by inclusion. The \(v\)-closed prime ideals are exactly those prime ideals \(\mathfrak{A}\) of \(A_v\) with \(v^{-1}(0) \subseteq \mathfrak{A} \subseteq P_v\), and these are in 1-1 order inverting correspondence with the isolated subgroups of \(\Gamma\). If the isolated subgroup \(\Sigma\) corresponds with the \(v\)-closed prime ideal \(Q\), then with \(B = \{ x \in R \mid xQ \subseteq Q \}\), \((B, Q)\) is the valuation pair corresponding to the valuation induced by the homomorphism \(\Gamma \to \Gamma/\Sigma\).

**Proof.** Straightforward.

If \((v, \Gamma)\) and \((w, \Lambda)\) are two valuations on \(R\) and \(w = \phi \circ v\) where \(\phi\) is an order homomorphism of \(\Gamma\) onto \(\Lambda\), we say \(w\) dominates \(v\) and write \(w \succeq v\). Valuations \(v\) and \(v'\) are called dependent if there is a valuation \(w\) with \(w \succeq v\) and \(w \succeq v'\) and \(w(R) \neq \{ w(1), w(0) \}\); and they are called independent otherwise. Note that \(w \succeq v\) implies that \(v^{-1}(v(0)) = w^{-1}(w(0))\).

**Proposition 4.** Let \((v, \Gamma)\) and \((w, \Lambda)\) be valuations on \(R\). Then \(w \succeq v\) if and only if \(A_v \subseteq A_w\) and \(v^{-1}(v(0)) \subseteq P_w \subseteq P_v\).

Suppose \(w \succeq v\). Then \(P_w\) is a \(v\)-closed prime ideal of \(A_v\). If \(\Sigma\) is the isolated subgroup of \(\Gamma\) corresponding to \(P_w\), then \(\Lambda \cong \Gamma/\Sigma\) and there is a natural valuation \(((w, v), \Sigma)\) on \(A_v/P_w\) such that

\[
\begin{array}{ccc}
A_v/P_w & \xrightarrow{v} & \Sigma \\
\downarrow \eta & & \downarrow \eta \\
A_v/P_w & & (w, v)
\end{array}
\]

commutes, where \(\eta\) is the natural homomorphism.

Further, \((A_{(w, v)}, P_{(w, v)}) = (A_v/P_w, P_v/P_w)\). \((w, v)\) is called the induced valuation.

The proof in this and the following is essentially that used for fields.

**Proposition 5.** Let \((v_1, \Gamma_1)\) and \((v_2, \Gamma_2)\) be distinct dependent valua-
tions on R. Then there is a valuation v on R with v ≥ v₁ and v ≥ v₂ such that (v, v₁) and (v, v₂) are independent valuations on Aᵥ/Pᵥ.

Let R be an extension of a ring K, (v₀, Γ₀) a valuation on K. A valuation (v, Γ) on R is called an extension of (v₀, Γ₀) to R (or v an extension of v₀ to R) if there is an order isomorphism φ of Γ₀ into Γ such that v(x) = φ ◦ v₀(x) for all x ∈ K.

**Proposition 6.** Let R be an extension of K, (v₀, Γ₀) a valuation on K, (v, Γ) a valuation on R. Then the following are equivalent.

(i) (v, Γ) is an extension of (v₀, Γ₀).

(ii) Aᵥ₀ ⊆ Aᵥ, Pᵥ₀ = Aᵥ₀ ∩ Pᵥ and v|ᵥ is a valuation on K.

(iii) Aᵥ₀ ⊆ Aᵥ, Pᵥ₀ = Aᵥ₀ ∩ Pᵥ and v₀⁻¹(v₀(0)) ⊆ v⁻¹(v(0)).

**Proof.** Straightforward.

**Proposition 7.** Let R be an extension of K, (v₀, Γ₀) a valuation on K. Then (v₀, Γ₀) has extensions to R if and only if K ∩ ℝ = ℝ, where ℝ = v₀⁻¹(v₀(0)).

**Proof.** The "only if" follows from (iii) of (6). For the "if," one checks that if (A, P) is a valuation pair of R with Pᵥ₀ + ℝ ⊆ P, Aᵥ₀ ⊆ A and Pᵥ₀ = P ∩ Aᵥ₀ (use Zorn), then the valuation induced by (A, P) satisfies (iii) of (6).

In particular, if R is an integral extension of K, then every valuation (v₀, Γ₀) of K has extensions to R (since ℝ is a prime ideal of K). In this case, one can show if (v, Γ) is a valuation on R with Aᵥ₀ ⊆ Aᵥ and Pᵥ₀ = Aᵥ₀ ∩ Pᵥ, then (v, Γ) extends (v₀, Γ₀).

**Proposition 8.** Let R be an extension of K, (v₀, Γ₀), (w₀, Γ₀) be valuations on K, (v, Γ), (w, Γ) be valuations on R. Then

(i) If w₀ ≥ v₀, then v₀ has extensions to R if and only if w₀ has extensions to R.

(ii) If w ≥ v and v|ᵥ is a valuation on K, then so is w|ᵥ and w|ᵥ ≥ v|ᵥ.

(iii) If w₀ ≥ v₀ and v is an extension of v₀ to R, then the set of extensions w of w₀ to R such that w ≥ v is nonempty and linearly ordered. If the group part of Γ/Γ₀ is torsion, then there is a unique extension w of w₀ to R with w ≥ v.

(iv) If w₀ ≥ w₀, w ≥ v, w extends w₀ and v extends v₀, then the induced valuation (w, v) extends the induced valuation (w₀, v₀).

**Proof.** Everything is straightforward except perhaps the uniqueness in (iii). To check this, let w and w' be extensions of w₀ to R dominating v. Let H and H' be the isolated subgroups of Γ determined by w and w' and suppose Γ/Γ₀ is torsion. By symmetry, it suffices to
show that $\mathcal{H} \subseteq \mathcal{H}$. Let $\alpha \in \mathcal{H}$. Since $\Gamma/\Gamma_0$ is torsion, there is an integer $n$ with $\alpha^n \in \Gamma_0$, so $\alpha^n \in \mathcal{H} \cap \Gamma_0 = \mathcal{H}' \cap \Gamma_0$. If $\alpha > v(1)$, then $\alpha^n \geq \alpha \geq v(1)$ so that $\alpha \in \mathcal{H}$. If $\alpha \leq v(1)$, then $\alpha^n \leq \alpha \leq v(1)$ and $\alpha \in \mathcal{H}$.

We say that a set $V$ of valuations on a ring $R$ has the inverse property if for each $x$ in $R$ there is an $x'$ in $R$ such that $v(xx') = v(1)$ whenever $v \in V$ and $v(x) \neq 0$.

**Proposition 9.** Let $V$ be a set of valuations on $R$ which has the inverse property, $V'$ a set of valuations on $R$ such that for each $v' \in V'$ there is a $v \in V$ with $v' \geq v$. Then $V \cup V'$ has the inverse property.

**Proposition 10.** Let $V$ be a set of valuations on $R$ with the inverse property, $w$ a valuation on $R$ such that $w \geq v$ for all $v \in V$. Then $\{(w, v) \mid v \in V\}$ has the inverse property.

The proofs of 9 and 10 are straightforward.

**Proposition 11.** Let $(v_0, \Gamma_0)$ be a valuation on $K$, $R$ an extension of $K$ and suppose the set $V$ of extensions of $v_0$ to $R$ is nonempty. Let $\mathfrak{A} = V_0^{-1}(v_0(0))$. Then if $x + R\mathfrak{A}$ is algebraic over $K/\mathfrak{A}$, there is an $x' \in R$ such that $v(xx') = v(1)$ whenever $(v, \Gamma) \in V$ and $v(x) \neq 0$. Also $v(x)\Gamma_0$ is torsion in $\Gamma/\Gamma_0$.

**Proof.** $v(x)\Gamma_0$ torsion is proven exactly as in the field case and the existence of $x'$ is a variation on this proof.

For $K$ a domain, $R$ an extension of $K$, the rank of $R$ over $K$ is the dimension of $R_K \setminus \alpha$ over the quotient field of $K$. If $v_0$ is a valuation on $K$ and an ideal $I$ is torsion in $(\Gamma/\Gamma_0)$.

**Corollary 11.** Let $(v_0, \Gamma_0)$ be a valuation on $K$, $R$ an extension of $K$ and suppose the set $V$ of extensions of $v_0$ to $R$ is nonempty. Then if $\operatorname{rank}_{v_0} R$ is finite, $V$ has the inverse property and $\Gamma/\Gamma_0$ is torsion for all $(v, \Gamma) \in V$.

**Proof.** Every element of $R/R\mathfrak{A}$ is algebraic over $K/\mathfrak{A}$.

**Convention.** For the remainder of this paper $R$ is an extension of $K$, $(v_0, \Gamma_0)$ is a valuation on $K$ such that the set of extensions $V$ of $v_0$ to $R$ is nonempty and $\operatorname{rank}_{v_0} R < \infty$. It is also assumed that $P_{v_0}$ is not an ideal of $K$ (the results hold with minor modifications when $P_{v_0}$ is an ideal) in order to hold down bookwork. For $x \in R$, $x'$ is an element with $v(xx') = v(1)$ whenever $v \in V$.

**Proposition 12.** Let $(v_1, \Gamma_1), (v_2, \Gamma_2)$ be distinct elements of $V$. Then $P_{v_1} \subseteq P_{v_2}$. 
Proof. One checks that \( A_{s_i} \neq A_{s_1} \) by showing that

\[
P_{s_i} = \{ x \in A_{s_i} \mid \exists y \in A_{s_1} \text{ such that } xy \in A_{s_1} \}.
\]

Then consider:

**Case I.** \( A_{s_1} \setminus A_{s_1} \neq \emptyset \). Let \( y \in A_{s_1} \setminus A_{s_1} \), so \( v_1(y) \leq v_1(1), v_1(1) < v_2(y) \). Since \( \Gamma_i/\Gamma_0 \) is torsion, there is an integer \( n > 0 \), \( a \in K \) with \( v_2(y^n) = v_0(a) > v_0(1) \). Then \( v_2(y) = v_2(y^{n+1}a') > v_2(1) \) while \( v_1(y^{n+1}a') = v_1(a') \leq v_1(1) \). Thus \( y^{n+1}a' \in P_{s_i} \setminus P_{s_1} \).

**Case II.** \( A_{s_2} \setminus A_{s_1} \neq \emptyset \). By Case I there is a \( y \in R \) with \( v_1(y) > v_1(1) \) and \( v_2(1) > v_2(y) \). Then \( v_2(1+y) = v_1(1+y) \) while \( v_2(1+y) = v_2(1) \), thus \( v_1((1+y)') < v_1(1) \) while \( v_2((1+y)') = v_2(1) \). That is \( (1+y)' \in P_{s_1} \setminus P_{s_1} \).

**Proposition 13.** Let \( v_1, v_2, \ldots, v_n \) be distinct elements of \( V \). Then there is an \( x \in R \) with \( v_i(x) > v_i(1) \) and \( v_i(1) > v_i(x) \) for \( i > 1 \).

Proof. By induction on \( n \). If \( v_i(x) \geq v_i(1) \) and \( v_i(1) > v_i(x) \) for \( 1 < i \leq r \) (as in \( r = 2 \)), then since \( \Gamma_i/\Gamma_0 \) is torsion, there is an \( a \in K \) with \( v_0(a) = v_0(0) \) and a positive integer \( m \) such that \( v_1(x) > v_1(a) \) for \( 1 < i \leq r \). Then \( v_1(x^m) > v_1(1) \) and \( v_1(x^m) < v_1(1) \), \( 1 < i \leq r \).

Now assuming 13 holds for \( r = n - 1 \), let \( j = 2, 3 \) and choose \( y_j \in R \) with \( v_j(y_j) > v_j(1) \) and \( v_i(y_j) = v_i(1) \) for \( i > 1 \) and \( i \neq j \). If \( v_j(y_j) \leq v_j(1) \) let \( x_i = y_j \), otherwise let \( x_i = (1+y_j)'y_j \). One checks that \( v_i(x_1x_2) > v_i(1) \) and \( v_i(x_1,x_2) < v_i(1) \) for \( i > 1 \).

**Corollary 14.** Let \( v_1, v_2, \ldots, v_n \) be distinct elements of \( V \), \( a \in R \). Then there is a \( b \in R \) with \( v_i(b) = v_i(a) \) if \( v_i(a) \geq v_i(1) \) and \( v_i(b) = v_i(1) \) otherwise.

Proof. Use 13.

**Proposition 15.** Let \( v_1, v_2, \ldots, v_n \) be pairwise independent elements of \( V \), \( \alpha_i \) nonzero elements of \( \Gamma_i \), \( i = 1, 2 \ldots n \). Then there is an \( x \in R \) with \( v_i(x) = \alpha_i, i = 1, 2 \ldots n \).

Proof. Essentially the same as for fields, except that one uses the torsion and inverse properties as in 12 and 13 where one would ordinarily take an inverse.

For \( (v, \Gamma) \in V \), we let \( f \) be the rank of \( A_\alpha/P_\alpha \) over \( A_\alpha/P_\alpha \) and \( e \) be the index of \( \Gamma_0 \) in \( \Gamma \).

Using 15 and a long argument identical to the proof in [5] of the comparable theorem, one can establish

**Proposition 16.** Let \( v_1, v_2, \ldots, v_n \) be distinct elements of \( V \). Then \( \sum_{i=1}^n \epsilon_i f e_i \leq \text{rank}_{e_i} R \). In particular \( V \) is finite.
Remark. Using the definition and properties of Galois extensions as found in [2] and [4], one can prove the analogues of the results for Galois extensions of fields as found in [S], including \( efg\pi^d = |G| \), where \( |G| \) is the number of elements in a Galois group \( G \) for \( R \) over \( K \); \( e = e_v, f = f_v \) for any \( v \in V \); \( g \) is the number of elements in \( V \); \( d \) is a nonnegative integer; and \( \pi = 1 \) if \( \text{char } K = 0 \), \( \pi = \text{char } K \) otherwise.

Bibliography


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