VALUATIONS ON A COMMUTATIVE RING[1]

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By a valuation on a commutative ring R, we mean a pair (v, Γ), where Γ is an ordered (mult) group with a zero adjoined and v is a map of R *onto* Γ satisfying

(1) $v(xy)=v(x)v(y)$ for all $x, y\in R$,

(2) $v(x+y)\leqq\max\{v(x), v(y)\}$ for all $x, y\in R$.

The pair will usually be denoted simply by v.

The definition without the "onto" requirement is standard, but this requirement is crucial in obtaining our analogues of classical theorems on valuations on a field and results already used in [3].

Throughout this paper, "ring" means "commutative ring with 1" and subrings always contain that 1. Other conventions and notation used should be clear from context.

PROPOSITION 1. *Let A be a subring of R, P a prime ideal of A. Then the following are equivalent.*

(i) *For each subring B of R, and ideal Q of B with $A\subset B$ and $Q\cap A=P$, one has $A=B$.*

(ii) *For $x\in R\backslash A$, there is an $x'\in P$ with $xx'\in A\backslash P$.*

(iii) *There is a valuation v on R with $A=\{x\in R \mid v(x)\leqq v(1)\}$ and $P=\{x\in R \mid v(x)<v(1)\}$.*

PROOF. (i)$\Rightarrow$(ii) is straightforward using the "trick" used in [5] to show that a valuation on a subfield can be extended to an overfield.

(ii)$\Rightarrow$(iii). Write $x\sim y$ when $\{z\in R \mid xz\in P\}=\{z\in R \mid yz\in P\}$. One checks that $\sim$ is an equivalence relation on R. Let $v(x)=\{y \mid x\sim y\}$ and $\Gamma=\{v(x) \mid x\in R\}$. One checks: $v(x)<v(y)$ whenever $\exists z\in R$ with $zx\in P$ and $zy\notin P$ and $v(x)v(y)=v(xy)$ are well defined and (v, Γ) is a valuation on R with the required properties.

(iii)$\Rightarrow$(i). Using standard techniques for fields, one shows $A=\{x \mid v(x)\leqq v(1)\}$ is a subring of R and $\{x \mid v(x)<v(1)\}=P$ is a prime ideal of A. If $x\notin A$ then $v(x)^{-1}=v(x')$ for some $x'\in P$, and since $xx'\in A\backslash P$, the proposition follows.

We call pairs satisfying (i), (ii) and (iii) valuation pairs. Note they are the subject of exercise 7, Chapter 6 of [1].

PROPOSITION 2. *Two valuations (v, Γ) and (w, Λ) determine the same*

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valuation pair if and only if there is an order isomorphism $\phi: \Gamma \rightarrow \Lambda$ *such that* $w = \phi \circ v$.

Proof. The same as for fields.

Henceforth we speak of *the valuation* determined by a valuation pair. Given a valuation (v, Γ) on R we let

$$A_v = \{x \in R \mid v(x) \leqq v(1)\} \quad \text{and} \quad P_v = \{x \in R \mid v(x) < v(1)\}.$$

Let (v, Γ) be a valuation on R. A v-closed ideal $\mathfrak{A}$ of A_v is an ideal such that $x \in \mathfrak{A}$, $v(y) \leqq v(x)$ implies $y \in \mathfrak{A}$.

Proposition 3. *The v-closed ideals of A_v are linearly ordered by inclusion. The v-closed prime ideals are exactly those prime ideals $\mathfrak{A}$ of A_v with $v^{-1}(0) \subseteq \mathfrak{A} \subseteq P_v$ and these are in* 1-1 *order inverting correspondence with the isolated subgroups of* Γ. *If the isolated subgroup Σ corresponds with the v-closed prime ideal Q, then with $B = \{x \in R \mid xQ \subset Q\}$, (B, Q) is the valuation pair corresponding to the valuation induced by the homomorphism* $\Gamma \rightarrow \Gamma/\Sigma$.

Proof. Straightforward.

If (v, Γ) and (w, Λ) are two valuations on R and $w = \phi \circ v$ where ϕ is an order homomorphism of Γ onto Λ, we say w dominates v and write $w \geqq v$. Valuations v and v' are called dependent if there is a valuation w with $w \geqq v$ and $w \geqq v'$ and $w(R) \neq \{w(1), w(0)\}$; and they are called independent otherwise. Note that $w \geqq v$ implies that $v^{-1}(v(0)) = w^{-1}(w(0))$.

Proposition 4. *Let (v, Γ) and (w, Λ) be valuations on R. Then $w \geqq v$ if and only if $A_v \subseteq A_w$ and $v^{-1}(v(0)) \subseteq P_w \subseteq P_v$.*

Suppose $w \geqq v$. Then P_w is a v-closed prime ideal of A_v. If Σ is the isolated subgroup of Γ corresponding to P_w, then $\Lambda \simeq \Gamma/\Sigma$ and there is a natural valuation $((w, v), \Sigma)$ on A_w/P_w such that

$$\begin{array}{ccc} A_w \backslash P_w & \xrightarrow{\ v\ } & \Sigma \\ \downarrow \eta & \nearrow & (w, v) \\ A_w/P_w & & \end{array}$$

commutes, where η is the natural homomorphism.

Further, $(A_{(w,v)}, P_{(w,v)}) = (A_v/P_w, P_v/P_w)$. (w, v) is called the induced valuation.

The proof in this and the following is essentially that used for fields.

Proposition 5. *Let (v_1, Γ_1) and (v_2, Γ_2) be distinct dependent valua-*

tions on R. Then there is a valuation v on R with $v \geqq v_1$ and $v \geqq v_2$ such that (v, v_1) and (v, v_2) are independent valuations on A_v/P_v.

Let R be an extension of a ring K, (v_0, Γ_0) a valuation on K. A valuation (v, Γ) on R is called an extension of (v_0, Γ_0) to R (or v an extension of v_0 to R) if there is an order isomorphism ϕ of Γ_0 into Γ such that $v(x) = \phi \circ v_0(x)$ for all $x \in K$.

PROPOSITION 6. *Let R be an extension of K, (v_0, Γ_0) a valuation on K, (v, Γ) a valuation on R. Then the following are equivalent.*

(i) *(v, Γ) is an extension of (v_0, Γ_0).*

(ii) *$A_{v_0} \subseteq A_v$, $P_{v_0} = A_{v_0} \cap P_v$ and $v|_K$ is a valuation on K.*

(iii) *$A_{v_0} \subseteq A_v$, $P_{v_0} = A_{v_0} \cap P_v$ and $v_0^{-1}(v_0(0)) \subseteq v^{-1}(v(0))$.*

PROOF. Straightforward.

PROPOSITION 7. *Let R be an extention of K, (v_0, Γ_0) a valuation on K. Then (v_0, Γ_0) has extensions to R if and only if $K \cap R\mathfrak{A} = \mathfrak{A}$, where $\mathfrak{A} = v_0^{-1}(v_0(0))$.*

PROOF. The "only if" follows from (iii) of (6). For the "if," one checks that if (A, P) is a valuation pair of R with $P_{v_0} + R\mathfrak{A} \subset P$, $A_{v_0} \subset A$ and $P_{v_0} = P \cap A_{v_0}$ (use Zorn), then the valuation induced by (A, P) satisfies (iii) of (6).

In particular, if R is an integral extension of K, then every valuation (v_0, Γ_0) of K has extensions to R (since $\mathfrak{A}$ is a prime ideal of K). In this case, one can show if (v, Γ) is a valuation on R with $A_{v_0} \subset A_v$ and $P_{v_0} = A_{v_0} \cap P_v$, then (v, Γ) extends (v_0, Γ_0).

PROPOSITION 8. *Let R be an extension of K, (v_0, Γ_0), (w_0, Γ_0) be valuations on K, (v, Γ), (w, Γ) be valuations on R. Then*

(i) *If $w_0 \geqq v_0$, then v_0 has extensions to R if and only if w_0 has extensions to R.*

(ii) *If $w \geqq v$ and $v|_K$ is a valuation on K, then so is $w|_K$ and $w|_K \geqq v|_K$.*

(iii) *If $w_0 \geqq v_0$ and v is an extension of v_0 to R, then the set of extensions w of w_0 to R such that $w \geqq v$ is nonempty and linearly ordered. If the group part of Γ/Γ_0 is torsion, then there is a unique extension w of w_0 to R with $w \geqq v$.*

(iv) *If $w_0 \geqq v_0$, $w \geqq v$, w extends w_0 and v extends v_0, then the induced valuation (w, v) extends the induced valuation (w_0, v_0).*

PROOF. Everything is straightforward except perhaps the uniqueness in (iii). To check this, let w and w' be extensions of w_0 to R dominating v. Let H and H' be the isolated subgroups of Γ determined by w and w' and suppose Γ/Γ_0 is torsion. By symmetry, it suffices to show that $H' \subseteq H$. Let $\alpha \in H'$. Since Γ/Γ_0 is torsion, there is an integer n with $\alpha^n \in \Gamma_0$, so $\alpha^n \in H$ $(H \cap \Gamma_0 = H' \cap \Gamma_0)$. If $\alpha > v(1)$, then $\alpha^n \geqq \alpha \geqq v(1)$ so that $\alpha \in H$. If $\alpha \leqq v(1)$, then $\alpha^n \leqq \alpha \leqq v(1)$ and $\alpha \in H$.

We say that a set V of valuations on a ring R has the inverse property if for each x in R there is an x' in R such that $v(xx') = v(1)$ whenever $v \in V$ and $v(x) \neq v(0)$.

PROPOSITION 9. *Let V be a set of valuations on R which has the inverse property, V' a set of valuations on R such that for each $v' \in V'$ there is a $v \in V$ with $v' \geqq v$. Then $V \cup V'$ has the inverse property.*

PROPOSITION 10. *Let V be a set of valuations on R with the inverse property, w a valuation on R such that $w \geqq v$ for all $v \in V$. Then $\{(w, v) \mid v \in V\}$ has the inverse property.*

The proofs of 9 and 10 are straightforward.

PROPOSITION 11. *Let (v_0, Γ_0) be a valuation on K, R an extension of K and suppose the set V of extensions of v_0 to R is nonempty. Let $\mathfrak{A} = V_0^{-1}(v_0(0))$. Then if $x + R\mathfrak{A}$ is algebraic over $K/\mathfrak{A}$, there is an $x' \in R$ such that $v(xx') = v(1)$ whenever $(v, \Gamma) \in V$ and $v(x) \neq 0$. Also $v(x)\Gamma_0$ is torsion in Γ/Γ_0.*

PROOF. $v(x)\Gamma_0$ torsion is proven exactly as in the field case and the existence of x' is a variation on this proof.

For K a domain, R an extension of K, the rank of R over K is the dimension of $R_{K \backslash \{0\}}$ over the quotient field of K. If v_0 is a valuation on K with extensions to R, $\operatorname{rank}_{v_0} R$ is the rank of $R/R\mathfrak{A}$ over $K/\mathfrak{A}$, where $\mathfrak{A} = v_0^{-1}(v_0(0))$.

COROLLARY 11. *Let (v_0, Γ_0) be a valuation on K, R an extension of K and suppose the set V of extensions of v_0 to R is nonempty. Then if* $\operatorname{rank}_{v_0} R$ *is finite, V has the inverse property and Γ/Γ_0 is torsion for all $(v, \Gamma) \in V$.*

PROOF. Every element of $R/R\mathfrak{A}$ is algebraic over $K/\mathfrak{A}$.

CONVENTION. For the remainder of this paper R is an extension of K, (v_0, Γ_0) is a valuation on K such that the set of extensions V of v_0 to R is nonempty and $\operatorname{rank}_{v_0} R < \infty$. It is also assumed that P_{v_0} is not an ideal of K (the results hold with minor modifications when P_{v_0} is an ideal) in order to hold down bookwork. For $x \in R$, x' is an element with $v(xx') = v(1)$ whenever $v \in V$.

PROPOSITION 12. *Let (v_1, Γ_1), (v_2, Γ_2) be distinct elements of V. Then $P_{v_1} \nsubseteq P_{v_2}$.*

PROOF. One checks that $A_{v_1} \neq A_{v_2}$ by showing that

$$P_{v_1} = \{x \in A_{v_1} \mid \exists y \notin A_{v_1} \text{ such that } xy \in A_{v_1}\}.$$

Then consider:

CASE I. $A_{v_1} \backslash A_{v_2} \neq \emptyset$. Let $y \in A_{v_1} \backslash A_{v_2}$, so $v_1(y) \leqq v_1(1)$, $v_2(1) < v_2(y)$. Since Γ_i/Γ_0 is torsion, there is an integer $n > 0$, $a \in K$ with $v_2(y^n) = v_0(a) > v_0(1)$. Then $v_2(y) = v_2(y^{n+1}a') > v_2(1)$ while $v_1(y^{n+1}a') = v_1(y^{n+1})v_1(a') \leqq v_1(a') < v_1(1)$. Thus $y^{n+1}a' \in P_{v_1} \backslash P_{v_2}$.

CASE II. $A_{v_2} \backslash A_{v_1} \neq \emptyset$. By Case I there is a $y \in R$ with $v_1(y) > v_1(1)$ and $v_2(1) > v_2(y)$. Then $v_1(1+y) = v_1(y)$ while $v_2(1+y) = v_2(1)$, thus $v_1((1+y)') < v_1(1)$ while $v_2((1+y)') = v_2(1)$. That is $(1+y)' \in P_{v_1} \backslash P_{v_2}$.

PROPOSITION 13. *Let $v_1, v_2 \cdots v_n$ be distinct elements of V. Then there is an $x \in R$ with $v_1(x) > v_1(1)$ and $v_i(1) > v_i(x)$ for $i > 1$.*

PROOF. By induction on n. If $v_1(x) \geqq v_1(1)$ and $v_i(1) > v_i(x)$ for $1 < i \leqq r$ (as in $r = 2$), then since Γ_i/Γ_0 is torsion, there is an $a \in K$ with $v_0(a) \neq v_0(0)$ and a positive integer m such that $v_i(1) > v_i(a) > v_i(x^m)$ for $1 < i \leqq r$. Then $v_1(x^m a') > v_1(1)$ and $v_i(x^m a') < v_i(1)$, $1 < i \leqq r$.

Now assuming 13 holds for $r = n-1$, let $j = 2, 3$ and choose $y_j \in R$ with $v_1(y_j) > v_1(1)$, $v_i(1) > v_i(y_j)$ for $i > 1$ and $i \neq j$. If $v_j(y_j) \leqq v_j(1)$ let $x_j = y_j$, otherwise let $x_j = (1+y_j)'y_j$. One checks that $v_1(x_1x_2) \geqq v_1(1)$ and $v_i(x_1x_2) < v_i(1)$ for $i > 1$.

COROLLARY 14. *Let $v_1, v_2 \cdots v_n$ be distinct elements of V, $a \in R$. Then there is a $b \in R$ with $v_i(b) = v_i(a)$ if $v_i(a) \geqq v_i(1)$ and $v_i(b) = v_i(1)$ otherwise.*

PROOF. Use 13.

PROPOSITION 15. *Let $v_1, v_2 \cdots v_n$ be pairwise independent elements of V, α_i nonzero elements of Γ_i, $i = 1, 2 \cdots n$. Then there is an $x \in R$ with $v_i(x) = \alpha_i$, $i = 1, 2 \cdots n$.*

PROOF. Essentially the same as for fields, except that one uses the torsion and inverse properties as in 12 and 13 where one would ordinarily take an inverse.

For $(v, \Gamma) \in V$, we let f_v be the rank of A_v/P_v over A_{v_0}/P_{v_0} and e_v be the index of Γ_0 in Γ.

Using 15 and a long argument identical to the proof in [5] of the comparable theorem, one can establish

PROPOSITION 16. *Let $v_1, v_2 \cdots v_n$ be distinct elements of V. Then $\sum_{i=1}^n e_{v_i} f_{v_i} \leqq \operatorname{rank}_{v_0} R$. In particular V is finite.*

REMARK. Using the definition and properties of Galois extensions as found in [**2**] and [**4**], one can prove the analogues of the results for Galois extensions of fields as found in [**5**], including "$efg\pi^d = |G|$," where $|G|$ is the number of elements in a Galois group G for R over K; $e = e_v$, $f = f_v$ for any $v \in V$; g is the number of elements in V; d is a nonnegative integer; and $\pi = 1$ if char $K = 0$, $\pi =$ char K otherwise.

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