NOTE ON QF-1 ALGEBRAS[1]

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1. **Introduction.** If M is an R-module and $E_1 = \mathrm{Hom}_R(M, M)$ its endomorphism ring, then M can be considered as an E_1 module in a natural way. Denote by E_2 the ring $\mathrm{Hom}_{E_1}(M, M)$. Then there is a ring homomorphism $\phi: R \rightarrow E_2$, since multiplication by an element $r \in R$ causes an E_1-homomorphism of M. Following [4], we shall say that M is *balanced* if ϕ is an epimorphism.

In [8], Thrall noted that every faithful module over a Quasi-Frobenius algebra is balanced. He then defined QF-1 algebras as algebras having this property that all faithful modules are balanced. He showed by example that the class of QF-1 algebras is more general than the class of Quasi-Frobenius algebras.

My former student Denis Floyd in studying QF-1 algebras [5] noted that if the algebra had certain kinds of indecomposable faithful modules with large composition length then it was not QF-1. This led him to the following conjecture: If A is a QF-1 algebra then there exists n such that if M is a faithful indecomposable A-module then the composition length of M is less than n.

We shall say that an algebra with such a bound on the composition lengths of faithful indecomposable modules is of *bounded faithful module type*. In support of Floyd's conjecture, one can show that Quasi-Frobenius (and QF-3) algebras [9] are of bounded faithful module type. There are a number of papers concerned with algebras of bounded or unbounded module type [1], [2], [5], [6], but these are concerned with arbitrary modules not just faithful ones.

In [2], Spencer Dickson introduced the concept of an indecomposable module having a large kernel. The A-module M has a *large kernel* if every nilpotent element of $E_1 = \mathrm{Hom}_A(M, M)$ annihilates the socle of M. The algebra A is said to have large kernels if every indecomposable A-module of finite composition length has this property.

In this note, we shall prove Floyd's conjecture under the additional assumption that faithful indecomposable A-modules have large kernels.

We shall make the following standing assumptions: we only consider finitely generated modules over an algebra A and A is a finite

Received by the editors September 21, 1967.

[1] Research supported in part by NSF Grant GP-6545.

dimensional algebra over field K. Although all the concepts we use apply to a more general situation (say, rings with minimum condition) our proofs rely on vector space dimension arguments so we restrict consideration to algebras.

2. **Computing** E_2. For the A-module M we use the notation $E_1 = \mathrm{Hom}_A(M, M)$ and $E_2 = \mathrm{Hom}_{E_1}(M, M)$. We denote by $S(M)$ the A-socle of M, the sum of all the simple A-submodules of M. In the following lemma we obtain a lower bound for the size of E_2 in terms of $S(M)$. Note that we need not assume that M is faithful in the lemma.

LEMMA. *If M is an indecomposable A-module having a large kernel then* $[E_2:K] \geqq [S(M):K]$.

PROOF. We first consider the case that $M = S(M)$. In this case, since M is A-indecomposable and A-semisimple, M must be A-simple. It follows that $E_1 = D$ a division algebra and E_2 is the D-endomorphism ring of $S(M)$, that is, a total matrix ring over D. It follows that $[E_2:D] \geqq [S(M):D]$ and hence $[E_2:K] \geqq [S(M):K]$.

In the following we can assume that $S(M) \neq M$. Since $S(M)$ is an E_1 submodule of M we can find a maximal E_1 submodule T of M such that $S(M) \subseteq T \subset M$.

Since M is A-indecomposable, E_1 has no nontrivial idempotents. It follows that $E_1/\mathrm{Rad}\, E_1$ is a division algebra D, (using the Wedderburn theorem [7]). We know then that simple E_1 modules (hence D-modules) are of dimension 1 over D any two such simple E_1 modules are E_1-isomorphic to D.

Now, we use the assumption that M has large kernels, that is, the nilpotent elements of E_1 anihilate $S(M)$. This means that $S(M)$ is a D-module and is therefore a direct sum of $[S(M):D]$ copies of D.

Finally, we put all the above information together to obtain our estimate on the size of E_2. In the first place we have

$$\mathrm{Hom}_{E_1}\ (M/T,\ S(M)) = \mathrm{Hom}_D\ (M/T,\ S(M))$$

and the dimension of both over D is $[S(M): D]$ because M/T is a simple D module and $S(M)$ the sum of $[S(M): D]$ copies of D. Then computing dimension over K we see that $[\mathrm{Hom}_D(M/T,\ S(M)):K] = [S(M):D][D:K] = [S(M):K]$.

Now note that $\mathrm{Hom}_{E_1}\ (M/T,\ S(M)) \subseteq \mathrm{Hom}_{E_1}\ (M,\ M) = E_2$, because E_1 homomorphism of M/T to $S(M)$ can be lifted to E_1 homomorphism of M to $S(M) \subseteq M$. It now follows that $[E_2:K] \geqq [S(M):K]$.

We can now prove the following theorem:

THEOREM. *If A is a* QF-1 *algebra having large kernels then A is of bounded faithful module type.*

PROOF. We construct a contrapositive proof using the lemma. Suppose that A has faithful indecomposable modules of arbitrarily large finite composition length. Let n_0 be the dimension of A over K, $[A:K]=n_0$.

If S is any simple A-module then the dimension of $Q(S)$, the minimal injective for S is less than or equal to n_0. This follows from the fact that $Q(S)$ is a direct summand of $\mathrm{Hom}_K(A, K)$ which has dimension n_0. If M is any A-module and $S(M)$ its socle then we have

$$(*) \qquad S(M) \subseteq M \subseteq Q(M) = Q(S(M)) = \oplus \sum_{1}^{t} Q(S_i)$$

where $Q(M)$ is the minimal injective of M and $S(M)=\oplus\sum_1^t S_i$. This follows from the fact that $S(M)$ is essential in M and $Q(M)$; see [**3**].

Now choose M indecomposable, faithful such that $[M:K]>n_0^2$. From the condition (*) and the inequality $[Q(S):K]\leqq n_0$, we see that the number t of simple summands in $S(M)$ satisfies $t>n_0$. Therefore $[S(M):K]>n_0$. Now by applying the lemma we see $[E_2:K]\geqq[S(M):K]>n_0=[A:K]$. It follows that the homomorphism from A to E_2 cannot be an epimorphism and A is not QF-1. Remark: The hypothesis of large kernels appears to be a strong one. Since the usual methods for constructing large indecomposable modules seem to give modules lacking large kernels, let us hazard the following

CONJECTURE. If indecomposable A-modules have large kernels then A has (up to isomorphism) only a finite number of indecomposable modules.

This conjecture would, of course, immediately imply the theorem of this paper. It would also give several results in Section 2 of Dickson's paper [2].

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