FORMATIONS OF GROUPS AND π-DECOMPOSABILITY

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1. **Introduction.** Suppose π is a set of prime numbers, π' the complement of π in the set of all prime numbers, $\mathfrak{F}$ a saturated formation of groups (see [**3**]). $\mathrm{Hall}_\pi G$ shall mean the set of all Hall π-subgroups of a group G and $\mathfrak{F}_\pi$ denotes the set $\{G \mid G \cong G_\pi \times G_{\pi'},\ G_\pi \in \mathrm{Hall}_\pi G,\ G_{\pi'} \in \mathrm{Hall}_{\pi'} G,\ G_\pi \in \mathfrak{F}\}$. The purpose of this note is to show that $\mathfrak{F}_\pi$ is also a saturated formation. As an application we shall be able to state a partial generalization of a known result of Romanovskiĭ [**6**] and to identify the K_π-subgroups as covering subgroups of a certain formation. All groups considered in this note are finite and, if there is not made a special remark, solvable.

2. **Main result.** We now proceed to prove the following theorem.

THEOREM 1. *If $\mathfrak{F}$ is a saturated formation, then $\mathfrak{F}_\pi$ is also a saturated formation.*

PROOF. Let N be a minimal normal subgroup of $G \in \mathfrak{F}_\pi$, $G_\pi \in \mathrm{Hall}_\pi G'$ $G_{\pi'} \in \mathrm{Hall}_{\pi'} G$. Then N is either a π-group or a π'-group. Thus $G/N \cong (G_\pi \times G_{\pi'})/N \cong G_\pi N/N \times G_{\pi'} N/N$ and $G_\pi N/N \cong G_\pi / N \cap G_\pi \in \mathfrak{F}$. Therefore $G \in \mathfrak{F}_\pi$, $N \lhd G$ implies $G/N \in \mathfrak{F}_\pi$.

Now let U denote a subgroup of G. Then $U/G_\pi \cap U \cong UG_\pi/G_\pi \subset G/G_\pi$ which implies, that $U/G_\pi \cap U$ is a π'-group. Since $G_\pi \cap U$ is a π-group we conclude $G_\pi \cap U \in \mathrm{Hall}_\pi U$ and similarly we find $G_{\pi'} \cap U \in \mathrm{Hall}_{\pi'} U$, hence $U \cong (G_\pi \cap U) \times (G_{\pi'} \cap U)$. Further, if U and V are direct products of their Hall π-subgroups by their Hall π'-subgroups, so $U \times V$ is. Now let N_1, N_2 be normal subgroups of G such that $G/N_1 \cong G_\pi N_1/N_1 \times G_{\pi'} N_1/N_1$ and $G/N_2 \cong G_\pi N_2/N_2 \times G_{\pi'} N_2/N_2$. Then the considerations above imply $G/N_1 \cap N_2 \cong G_\pi(N_1 \cap N_2)/N_1 \cap N_2 \times G_{\pi'}(N_1 \cap N_2)/N_1 \cap N_2$ and if $G_\pi N_1/N_1 \in \mathfrak{F}$, $G_\pi N_2/N_2 \in \mathfrak{F}$, then $G_\pi/N_1 \cap G \in \mathfrak{F}$, $G_\pi/N_2 \cap G \in \mathfrak{F}$. Therefore $G_\pi(N_1 \cap N_2)/N_1 \cap N_2 \cong G_\pi/N_1 \cap N_2 \cap G_\pi \in \mathfrak{F}$.

To show that $\mathfrak{F}_\pi$ is saturated we have to prove that $G \notin \mathfrak{F}_\pi$, N minimal normal subgroup of G, $G/N \in \mathfrak{F}_\pi$ implies N complemented. We use induction on $|G|$. N is the only minimal normal subgroup of G such that $G/N \in \mathfrak{F}_\pi$. If there exists another minimal normal subgroup of G, N_1 say, then $G/N_1 \notin \mathfrak{F}_\pi$. We now take a normal subgroup U of G, maximal for the property $N_1 \subset U < NN_1$. Then $G/U \notin \mathfrak{F}_\pi$ (since otherwise $N = G^{\mathfrak{F}_\pi} \subset U$ whence $U = NN_1$) but $G/NN_1 \in \mathfrak{F}_\pi$. By

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induction NN_1/U is complemented, i.e. there exists $R\supset U$, such that $G=NN_1R=NR$, $NN_1\cap R=U$. Therefore $N\cap R=N\cap U=1$, as $U<NN_1$. Thus R is a complement of N. We now assume N to be the only minimal normal subgroup of G. Let N be a π-group. If $G_{\pi'}=1$ then G is a π-group, $G\notin\mathfrak{F}$, $G/N\in\mathfrak{F}$ whence N is complemented, as $\mathfrak{F}$ is saturated. If $G_{\pi'}>1$, then $G_{\pi'}N/N$ contains a minimal normal π'-subgroup of G/N. Using a theorem of Ore (see [5, Chapter 4, Theorem 7]), N is complemented. If N is a π'-group then $G_\pi N/N$ contains a minimal normal π-subgroup of G/N and the argument above shows N to be complemented. Therefore $\mathfrak{F}_\pi$ is saturated.

3. **Applications of Theorem 1.** We start with some definitions.

DEFINITION ([6]). Let $\mathfrak{N}$ be the formation of nilpotent groups. A group G is called *π-decomposable* if $G\in\mathfrak{N}_\pi$.

DEFINITION ([6]). L is called a *K_π-subgroup* of a group G if L is a π-decomposable, self-normalizing subgroup of G such that $\text{Hall}_{\pi'}G=\text{Hall}_{\pi'}L$.

REMARK. Romanovskiĭ [6] has proved the following theorem: Let G (not necessarily solvable) contain a Hall normal π-subgroup. Then G has at least one K_π-subgroup being abnormal in G and any two K_π-subgroups are conjugate. By a simple argument of Carter [2] one shows, under the assumption that G be solvable, the K_π-subgroups coincide with the $\mathfrak{N}_\pi$-covering subgroups of G.

But there holds a more general statement.

PROPOSITION 1. *Let G be a solvable group. If K_π-subgroups of G exist then they are $\mathfrak{N}_\pi$-covering subgroups. In particular they are conjugate.*

PROOF. Let L be a K_π-subgroup of G. If $L\subset H<G$, clearly L is a K_π-subgroup of H. It follows by induction on $|H|$ that L is an $\mathfrak{N}_\pi$-covering subgroup of H whence $LH^{\mathfrak{N}_\pi}=H$. There remains only to show $LG^{\mathfrak{N}_\pi}=G$. If $G^{\mathfrak{N}_\pi}=1$, then $G\in\mathfrak{N}_\pi$. Since L is self-normalizing we conclude that G is the $\mathfrak{N}_\pi$-covering subgroup. Now assume $G^{\mathfrak{N}_\pi}>1$ and choose N_0 minimal normal in G but lying in $G^{\mathfrak{N}_\pi}$. Then LN_0/N_0 lies in $\mathfrak{N}_\pi$ and contains a Hall π'-subgroup of G/N_0. Set $T=N_G(LN_0)$, so $T/N_0=N_{G/N_0}(LN_0/N_0)$. Suppose $LN_0<G$, then by the remarks above, L is an $\mathfrak{N}_\pi$-covering subgroup of LN_0. Since $\mathfrak{N}_\pi$ is saturated by Theorem 1, L is an intravariant subgroup (see [4], [7]) of LN_0. Then by a Frattini argument $T=N_T(L)N_0=LN_0$, since L is self-normalizing. This equation shows that LN_0/N_0 is a K_π-subgroup of G/N_0. By induction, LN_0/N_0 is an $\mathfrak{N}_\pi$-covering subgroup of G/N_0, so $(LN_0/N_0)(G/N_0)^{\mathfrak{N}_\pi}=G/N_0$ whence $LG^{\mathfrak{N}_\pi}=G$ since $N_0\subset G^{\mathfrak{N}_\pi}$. Thus we assume $G=LN_0$. Since $G/N_0\in\mathfrak{N}_\pi$, $G\notin\mathfrak{N}_\pi$, it follows by minimality of N_0 that $N_0=G^{\mathfrak{N}_\pi}$. Hence $LG^{\mathfrak{N}_\pi}=G$.

COROLLARY. *If a solvable group G contains a self-normalizing π'-Hall subgroup, then that subgroup is an $\mathfrak{N}_\pi$-covering subgroup of G.*

PROOF. Such a subgroup is obviously a K_π-subgroup of G and the result follows by Proposition 1.

LEMMA. *Let $\mathfrak{F}$ be a saturated formation. Set $\pi = \{p \mid p$ is a prime such that $Z_p \in \mathfrak{F}\}$. Then in any finite solvable group G, $\mathfrak{F}$-covering subgroups have π'-index in their normalizers. Conversely, if $\pi_1 = \{p \mid p$ is a prime divisor of $[N_G(F)\colon F]$ where G ranges over all finite solvable groups and F is an $\mathfrak{F}$-covering subgroup of $G\}$, then $Z_p \notin \mathfrak{F}$ for any $p \in \pi_1$, i.e. $\pi_1 = \pi'$.*

PROOF. Let F be an $\mathfrak{F}$-covering subgroup of G. Then F is also an $\mathfrak{F}$-covering subgroup of $N_G(F)$. If $N_G(F) < G$, the result follows by induction. Assume now $F \lhd G$. If $F > 1$, let N be a minimal normal subgroup of G lying in N. Then F/N is an $\mathfrak{F}$-covering subgroup of G/N and since $N_{G/N}(F/N) = N_G(F)/N$ the result follows by induction on $|G/N|$. Assume $F = 1$. If $G = 1$ there is nothing to prove. Suppose $G > 1$ and p is a prime divisor of $|G|$. Then G contains a subgroup $H \cong Z_p$. Since $F \subset H$, $FH^{\mathfrak{F}} = H = H^{\mathfrak{F}}$ whence $Z_p \notin \mathfrak{F}$. Thus $p \in \pi'$ and we have shown G is a π'-group.

To prove the converse it suffices to show $\pi' \subset \pi_1$, as $\pi_1 \subset \pi'$ follows by the first part of the proof. Choose $p \in \pi'$ and set $G = Z_p$. Since $G \notin \mathfrak{F}$ and is simple, $G = G^{\mathfrak{F}}$ and the $\mathfrak{F}$-covering subgroup of G is the identity group. Thus $p \in \pi_1$.

REMARK. This lemma generalizes a result of Gaschütz at the beginning of the final section of [3].

COROLLARY. *If $\pi = \{p \mid Z_p \in \mathfrak{F}\}$ where $\mathfrak{F}$ is a saturated formation, then $\mathfrak{F}$-covering subgroups of π-groups are self-normalizing.*

DEFINITION. Let $\mathfrak{F}$ be a formation containing all cyclic groups of order p for $p \in \pi$. A subgroup H of a π-solvable group G is called a *$K_\pi(\mathfrak{F})$-subgroup* of G iff (i) $H \in \mathfrak{F}_\pi$, (ii) H contains a π'-Hall subgroup of G, and (iii) $H = N_G(H)$.

REMARK. By this definition then $K_\pi(\mathfrak{N})$-subgroups are Romanovskiĭ's K_π-subgroups.

We now prove a partial generalization of Romanovskiĭ's theorem.

PROPOSITION 2. *Let G be a (not necessarily solvable) group containing a normal solvable π-Hall subgroup N, and let $\mathfrak{F}$ be any saturated formation which contains groups of order p for all primes $p \in \pi$. Then there exist $K_\pi(\mathfrak{F})$-subgroups of G.*

PROOF. Since N is solvable, N has a π'-Hall complement B by Schur-Zassenhaus. Let $C = C_N(B)$ and set $H = BC_0 = B \times C_0$ where

C_0 is an $\mathfrak{F}$-covering subgroup of the solvable group C. To prove that H is a $K_\pi(\mathfrak{F})$-subgroup it suffices to show $N_G(H)=H$. Since $G=BN$ and $B\subset N_G(H)$ we have $N_G(H)=BN_N(H)$. Now $C_0\subset N_N(H)$. If $xC_0\subset N_N(H)$ then $\{B, xC_0\}\subset H\cap N=C_0$. Thus B is a π'-group of operators stabilizing the coset xC_0 of the solvable π-subgroup $N_N(H)$. It follows from Theorem 1 of Glauberman [4] that xC_0 contains a fixed point of B, i.e., $xC_0\subset C$ whence $N_N(H)\subset C$. Thus $N_N(H)=N_C(H)\subset N_C(C_0)=C_0$, since C_0 is self-normalizing by the corollary of the preceding lemma. Thus $N_G(H)=BN_N(H)=B\times C_0=H$.

REMARK. $K_\pi(\mathfrak{F})$-subgroups are not conjugate, in general. Let, e.g., $\mathfrak{F}$ be the formation of π-groups. Then $G\in\mathfrak{F}_\pi$ iff G is a direct product of its π-Hall and π'-Hall subgroups, in particular, any π-group belongs to $\mathfrak{F}_\pi$. Let G be a π-group. Any self-normalizing subgroup of G is then a $K_\pi(\mathfrak{F})$-subgroup, and obviously these subgroups need not be conjugate.

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