AN OPERATOR ERGODIC THEOREM FOR SEQUENCES OF FUNCTIONS[1]

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1. **Introduction.** In 1940 P. T. Maker [7] showed that in Birkhoff's ergodic theorem images of a single function f can be replaced by images of functions forming a double sequence, dominated by an integrable function and converging to f. Here we obtain an analogous generalization of the Chacon-Ornstein ratio ergodic theorem for operators. It is also shown by example that dominated sequences in general cannot be replaced by uniformly integrable sequences.

In 1957 L. Breiman applied Maker's theorem to establish "an ergodic theorem of information theory," (see [2] and [5]). An analogous application of a variant of our result is given in [6].

2. **The theorem.** Let $(X, \mathfrak{A}, \mu)$ be a σ-finite measure space. We consider *semi-Markovian* operators: positive linear operators mapping L_1 into L_1. If the L_1 norm of T is less than or equal to one, T is called sub-Markovian. The aspects of the theory of *sub-Markovian* operators of interest for us are developed e.g. in [8]; the assumption made there that $\mu(X)=1$ is, for most purposes, inessential.

All relations below are to be understood modulo sets of measure zero. By L_1^+ we denote the class of nonnegative, not identically vanishing functions of L_1. The operator $I+T+T^2+\cdots$ is written T_∞. We let A be a measurable subset of X, such that on A the operator T is conservative, the ratio theorem holds and the limit is well behaved. More precisely, we assume the following conditions:

(c_A) $T_\infty g=\infty$ or 0 on A for each $g\in L_1^+$ and

(r_A) If $f\in L_1^+$, $g\in L_1^+$, then on the set $A\cap\{T_\infty g>0\}$

$$D(f, g) \overset{\text{def}}{=} \lim_{n\to\infty} \sum_{i=0}^{n-1} T^i f \Big/ \sum_{i=0}^{n-1} T^i g$$

exists and is finite. Further, if $F_k\in L_1$, $F_k\downarrow 0$ $(k\to\infty)$ on X, then for each $g\in L_1^+$, $D(F_k, g)\to 0$ on the set $A\cap\{T_\infty g>0\}$.

The conditions (c_A) and (r_A) are satisfied if T is a sub-Markovian operator and A is the conservative part C of the space. We only verify (r_A). If $\mu(X)=1$, then on the set $C\cap\{T_\infty g>0\}$ one has

$$D(F_k, g) = E(RF_k/\mathfrak{C})/E(Rg/\mathfrak{C})$$

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where $\mathcal{C}$ is the σ-field of invariant sets and the operator R adds to the function $f\cdot 1_C$ the total contribution of the dissipative part D, (see [4], [3], [8, p. 211]). Since R and $E(\cdot/\mathcal{C})$ are sub-Markovian operators, the last assertion in (r_A) follows from the monotone continuity theorem for such operators (see [8, p. 187]). If μ is a σ-finite measure, we let π be an equivalent probability measure and $p=d\pi/d\mu$, the Radon-Nikodým derivative of π with respect to μ. Now define an operator U by $Ug=(1/p)\cdot T(p\cdot g)$ where $g\in L_1(\pi)$, or, equivalently, $p\cdot g\in L_1(\mu)$. The passage to π and U leaves the ratios in (r_A) invariant; hence the σ-finite case reduces to the case $\mu(X)=1$. More generally, if T is a semi-Markovian operator satisfying the boundedness assumption (b_h), then the conditions (c_A) and (r_A) hold if A is the conservative part YC^h of the set Y^h (see [9]).

We now state our theorem.

THEOREM 1. *Assume that T is a semi-Markovian operator satisfying for some set A the conditions (c_A) and (r_A). Let f_{ni} and g_{ni}, n, $i=0$, $1, \cdots$, be double sequences of functions in L_1^+ such that* $\lim_{n,i} f_{ni}=f$, $\lim_{n,i} g_{ni}=g$ *and*

$$\sup_{n,i} f_{ni}\in L_1^+,\qquad \sup_{n,i} g_{ni}\in L_1^+. \tag{1}$$

Then on the set $A\cap\{T_\infty g>0\}$ one has

$$\lim_{n\to\infty}\frac{\sum_{i=0}^{n-1}T^i f_{ni}}{\sum_{i=0}^{n-1}T^i g_{ni}}=D(f,g). \tag{2}$$

PROOF. It suffices to show that

$$\limsup_{n\to\infty}\left|\frac{\sum_{i=0}^{n-1}T^i f_{ni}}{\sum_{i=0}^{n-1}T^i g_{ni}}-\frac{\sum_{i=0}^{n-1}T^i f}{\sum_{i=0}^{n-1}T^i g}\right|=0. \tag{3}$$

We have

$$\left|\frac{\sum_{i=0}^{n-1}T^i f_{ni}}{\sum_{i=0}^{n-1}T^i g}-\frac{\sum_{i=0}^{n-1}T^i f}{\sum_{i=0}^{n-1}T^i g}\right|\leqq\frac{\sum_{i=0}^{n-1}T^i\left|f_{ni}-f\right|}{\sum_{i=0}^{n-1}T^i g}. \tag{4}$$

For each fixed M, N let

$$F_{NM} = \sup_{n \geqq N, m \geqq M} |f_{nm} - f| .$$

Then for $n > N$

$$\frac{\sum_{i=0}^{n-1} T^i |f_{ni} - f|}{\sum_{i=0}^{n-1} T^i g} \leqq \frac{\sum_{i=0}^{M-1} T^i F_{00}}{\sum_{i=0}^{n-1} T^i g} + \frac{\sum_{i=M}^{n-1} T^i F_{NM}}{\sum_{i=0}^{n-1} T^i g} . \tag{5}$$

Therefore

$$\limsup_{n\to\infty} \left| \frac{\sum_{i=0}^{n-1} T^i f_{ni}}{\sum_{i=0}^{n-1} T^i g} - \frac{\sum_{i=0}^{n-1} T^i f}{\sum_{i=0}^{n-1} T^i g} \right| \tag{6}$$

$$\leqq \limsup_{n\to\infty} \frac{\sum_{i=0}^{M-1} T^i F_{00}}{\sum_{i=0}^{n-1} T^i g} + \limsup_{n\to\infty} \frac{\sum_{i=M}^{n-1} T^i F_{NM}}{\sum_{i=0}^{n-1} T^i g} . \tag{7}$$

By (c_A) the first term in (7) is zero on the set $A \cap \{T_\infty g > 0\}$ while by (r_A) the second term is $D(F_{NM}, g)$ on the same set. Since $F_{NM} \leqq F_{kk} \downarrow 0$ $(N, M \to \infty)$ where $k = \min(N, M)$, again by (r_A) we have that $D(F_{NM}, g) \to 0$ on $A \cap \{T_\infty g > 0\}$; hence the expression (6) is zero, which proves the particular case of (3) when all the functions g_{ni} equal g. (3) follows because

$$\frac{\sum_{i=0}^{n-1} T^i f_{ni}}{\sum_{i=0}^{n-1} T^i g_{ni}} = \frac{\sum_{i=0}^{n-1} T^i f_{ni}}{\sum_{i=0}^{n-1} T^i g} \cdot \frac{\sum_{i=0}^{n-1} T^i g}{\sum_{i=0}^{n-1} T^i g_{ni}} \tag{8}$$

and the last ratio converges to $1/D(g, g) = 1$. This completes the proof of the theorem.

REMARK. The proof shows that if the assumption (r_A) is weakened by replacing the limit D by the limit superior, one still has the following conclusion: on $A \cap \{T_\infty g > 0\}$

$$\limsup_n \frac{\sum_{i=0}^{n-1} T^i f_{ni}}{\sum_{i=0}^{n-1} T^i g} = \limsup_n \frac{\sum_{i=0}^{n-1} T^i f}{\sum_{i=0}^{n-1} T^i g}, \tag{9}$$

and an analogous equality holds for the limit inferior.

The following corollary, concerned with single sequences, may be considered as a generalization of Hopf's decomposition theorem (see [8, p. 196]). This theorem asserts that the space X decomposes into the conservative part C and the dissipative part D: for each $g \in L_1^+$, $T_\infty g = \infty$ on $C \cap \{g > 0\}$, $T_\infty g < \infty$ on D.

COROLLARY 1. *Let T be a sub-Markovian operator, let $g_i \to g$ as $i \to \infty$, $g_i \in L_1^+$, $\sup_i g_i \in L_1^+$. Then*

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} T^i g_i \tag{10}$$

is infinite on the set $C \cap \{g > 0\}$ and is finite on the set D.

PROOF. The assertion about the behavior on C follows from Theorem 1 applied with $A = C, f_{ni} = f > 0, f \in L_1, g_{ni} = g_i$. On the other hand, on D we have

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} T^i g_i \leqq \lim_{n\to\infty} \sum_{i=0}^{n-1} T^i \left(\sup_j g_j \right) \tag{11}$$

which is finite by Hopf's theorem.

For semi-Markovian operators the corollary remains valid with C replaced by YC^h, D replaced by YD^h.

3. **A counterexample.** Is it possible to replace the assumption (1) in Theorem 1 by the weaker assumption of uniform integrability of sequences? We particularize as follows. Let (v_i), (p_i), $i = 1, 2, \cdots$, be sequences of positive constants satisfying

$$\sum_i p_i = 1, \quad v_n p_n \to 0, \quad \sum_n v_n p_n / n = \infty. \tag{12}$$

Now let $(X, \mathcal{A}, \mu)$ be a probability space such that there is a measurable partition $\{A_i\}$ of X with $\mu(A_i) = p_i$, $i = 1, 2, \cdots$. Let $g_{ni} = 1$ for all n, i and let

$$\begin{aligned} f_{ni} = f_i &= v_i \quad && \text{on } A_i, \\ &= 0 \quad && \text{on } X - A_i. \end{aligned}$$

The sequence f_i converges to $f=0$. The uniform integrability in the presence of pointwise convergence to zero is equivalent with the convergence of the integrals $\int f_i\,d\mu = v_i p_i$ to $\int 0\,d\mu = 0$; thus (f_i) is uniformly integrable. Moreover, on A_n

$$\sup_n \sum_{i=1}^{n} \frac{f_i}{n} = \sup_n \frac{f_n}{n} = \frac{v_n}{n}; \tag{13}$$

hence

$$\int \left(\sup_n \sum_{i=1}^{n} f_i/n \right) d\mu = \sum_n v_n p_n/n = \infty. \tag{14}$$

By a theorem of Blackwell and Dubins [1], there is, on a suitable probability space $(X^*, \mathcal{A}^*, \mu^*)$, a sequence of functions (f_i^*) with the same joint distribution as (f_i), and a σ-field $\mathcal{C}$ such that

$$\mu^* \left\{ \frac{1}{n} \sum_{1}^{n} E(f_i \mid \mathcal{C}) \to 0 \right\} = 0. \tag{15}$$

We identify the starred and the nonstarred expressions. Let the operator T be the conditional expectation $E(\cdot \mid \mathcal{C})$. Then $T1=1$, $T^2=T$, T satisfies the assumptions (c_A) and (r_A) with $A=X$, but by (15) the equality (2) fails on the entire space X.

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