AN OPERATOR ERGODIC THEOREM FOR SEQUENCES OF FUNCTIONS

E. M. KLIMKO AND LOUIS SUCHESTON

1. Introduction. In 1940 P. T. Maker [7] showed that in Birkhoff's ergodic theorem images of a single function \( f \) can be replaced by images of functions forming a double sequence, dominated by an integrable function and converging to \( f \). Here we obtain an analogous generalization of the Chacon-Ornstein ratio ergodic theorem for operators. It is also shown by example that dominated sequences in general cannot be replaced by uniformly integrable sequences.

In 1957 L. Breiman applied Maker's theorem to establish "an ergodic theorem of information theory," (see [2] and [5]). An analogous application of a variant of our result is given in [6].

2. The theorem. Let \((X, \mathcal{A}, \mu)\) be a \( \sigma \)-finite measure space. We consider semi-Markovian operators: positive linear operators mapping \( L_1 \) into \( L_1 \). If the \( L_1 \) norm of \( T \) is less than or equal to one, \( T \) is called sub-Markovian. The aspects of the theory of sub-Markovian operators of interest for us are developed e.g. in [8]; the assumption made there that \( \mu(X) = 1 \) is, for most purposes, inessential.

All relations below are to be understood modulo sets of measure zero. By \( L^+_1 \) we denote the class of nonnegative, not identically vanishing functions of \( L_1 \). The operator \( I + T + T^2 + \cdots \) is written \( T^\infty \). We let \( A \) be a measurable subset of \( X \), such that on \( A \) the operator \( T \) is conservative, the ratio theorem holds and the limit is well behaved. More precisely, we assume the following conditions:

\( (c_A) \) \( T^\infty g = \infty \) or 0 on \( A \) for each \( g \in L^+_1 \) and
\( (r_A) \) If \( f \in L^+_1, g \in L^+_1 \), then on the set \( A \cap \{ T^\infty g > 0 \} \)

\[ D(f, g) = \lim_{n \to \infty} \sum_{i=0}^{n-1} T^i f / \sum_{i=0}^{n-1} T^i g \]

exists and is finite. Further, if \( F_k \in L_1, F_k \downarrow 0 \) \( (k \to \infty) \) on \( X \), then for each \( g \in L^+_1, D(F_k, g) \to 0 \) on the set \( A \cap \{ T^\infty g > 0 \} \).

The conditions \( (c_A) \) and \( (r_A) \) are satisfied if \( T \) is a sub-Markovian operator and \( A \) is the conservative part \( C \) of the space. We only verify \( (r_A) \). If \( \mu(X) = 1 \), then on the set \( C \cap \{ T^\infty g > 0 \} \) one has

\[ D(F_k, g) = E(RF_k/\mathcal{C})/E(Rg/\mathcal{C}) \]

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where $\mathcal{C}$ is the $\sigma$-field of invariant sets and the operator $R$ adds to the function $f \cdot 1_\mathcal{C}$ the total contribution of the dissipative part $D$, (see [4], [3], [8, p. 211]). Since $R$ and $E(\cdot /\mathcal{C})$ are sub-Markovian operators, the last assertion in $(r_A)$ follows from the monotone continuity theorem for such operators (see [8, p. 187]). If $\mu$ is a $\sigma$-finite measure, we let $\pi$ be an equivalent probability measure and $p = d\pi/d\mu$, the Radon-Nikodym derivative of $\pi$ with respect to $\mu$. Now define an operator $U$ by $Ug = (1/p) \cdot T(p \cdot g)$ where $g \in L_1(\pi)$, or, equivalently, $p \cdot g \in L_1(\mu)$. The passage to $\pi$ and $U$ leaves the ratios in $(r_A)$ invariant; hence the $\sigma$-finite case reduces to the case $\mu(X) = 1$. More generally, if $T$ is a semi-Markovian operator satisfying the boundedness assumption $(ba)$, then the conditions $(c_A)$ and $(r_A)$ hold if $A$ is the conservative part $YC^h$ of the set $Y^h$ (see [9]).

We now state our theorem.

**Theorem 1.** Assume that $T$ is a semi-Markovian operator satisfying for some set $A$ the conditions $(c_A)$ and $(r_A)$. Let $f_{n,i}$ and $g_{n,i}$, $n, i = 0, 1, \cdots$, be double sequences of functions in $L_1^+$ such that $\lim_{n,i} f_{n,i} = f$, $\lim_{n,i} g_{n,i} = g$ and

$$
\sup_{n,i} f_{n,i} \in L_1^+, \quad \sup_{n,i} g_{n,i} \in L_1^+.
$$

Then on the set $A \cap \{ T \geq 0 \}$ one has

$$
\lim_{n \to \infty} \frac{\sum_{i=0}^{n-1} T^i f_{n,i}}{\sum_{i=0}^{n-1} T^i g_{n,i}} = D(f, g).
$$

**Proof.** It suffices to show that

$$
\lim_{n \to \infty} \left| \frac{\sum_{i=0}^{n-1} T^i f_{n,i}}{\sum_{i=0}^{n-1} T^i g_{n,i}} - \frac{\sum_{i=0}^{n-1} T^i f}{\sum_{i=0}^{n-1} T^i g} \right| = 0.
$$

We have

$$
\frac{\sum_{i=0}^{n-1} T^i f_{n,i}}{\sum_{i=0}^{n-1} T^i g} - \frac{\sum_{i=0}^{n-1} T^i f}{\sum_{i=0}^{n-1} T^i g} \leq \frac{\sum_{i=0}^{n-1} T^i |f_{n,i} - f|}{\sum_{i=0}^{n-1} T^i g}.
$$
For each fixed $M, N$ let

$$F_{NM} = \sup_{n \geq N, m \geq M} |f_{nm} - f|.$$  

Then for $n > N$

$$\frac{\sum_{i=0}^{n-1} T^i |f_{ni} - f|}{\sum_{i=0}^{n-1} T^i g} \leq \frac{\sum_{i=0}^{M-1} T^i f_{00}}{\sum_{i=0}^{n-1} T^i g} + \frac{\sum_{i=M}^{n-1} T^i F_{NM}}{\sum_{i=0}^{n-1} T^i g}.$$  

Therefore

$$\limsup_{n \to \infty} \left[ \frac{\sum_{i=0}^{n-1} T^i f_{ni}}{\sum_{i=0}^{n-1} T^i g} - \frac{\sum_{i=0}^{n-1} T^i f}{\sum_{i=0}^{n-1} T^i g} \right] \leq \limsup_{n \to \infty} \frac{\sum_{i=0}^{M-1} T^i F_{00}}{\sum_{i=0}^{n-1} T^i g} + \limsup_{n \to \infty} \frac{\sum_{i=M}^{n-1} T^i F_{NM}}{\sum_{i=0}^{n-1} T^i g}.$$  

By (cA) the first term in (7) is zero on the set $A \cap \{ T_{ag} > 0 \}$ while by (rA) the second term is $D(F_{NM}, g)$ on the same set. Since $F_{NM} \leq F_{kk} \downarrow 0$ $(N, M \to \infty)$ where $k = \min(N, M)$, again by (rA) we have that $D(F_{NM}, g) \to 0$ on $A \cap \{ T_{ag} > 0 \}$; hence the expression (6) is zero, which proves the particular case of (3) when all the functions $g_{ni}$ equal $g$. (3) follows because

$$\sum_{i=0}^{n-1} T^i f_{ni} \over \sum_{i=0}^{n-1} T^i g = \sum_{i=0}^{n-1} T^i f_{ni} \over \sum_{i=0}^{n-1} T^i g \over \sum_{i=0}^{n-1} T^i g_{ni} = \sum_{i=0}^{n-1} T^i g_{ni} \over \sum_{i=0}^{n-1} T^i g \over \sum_{i=0}^{n-1} T^i g_{ni}$$  

and the last ratio converges to $1/D(g, g) = 1$. This completes the proof of the theorem.

**Remark.** The proof shows that if the assumption (rA) is weakened by replacing the limit $D$ by the limit superior, one still has the following conclusion: on $A \cap \{ T_{ag} > 0 \}$
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\[
\limsup_{n \to \infty} \frac{\sum_{i=0}^{n-1} T^i f_{n_i}}{\sum_{i=0}^{n-1} T^i g} = \limsup_{n \to \infty} \frac{\sum_{i=0}^{n-1} T^i f}{\sum_{i=0}^{n-1} T^i g},
\]

and an analogous equality holds for the limit inferior.

The following corollary, concerned with single sequences, may be considered as a generalization of Hopf's decomposition theorem (see [8, p. 196]). This theorem asserts that the space \( X \) decomposes into the conservative part \( C \) and the dissipative part \( D \): for each \( g \in L^+_1 \), \( T_n g = \infty \) on \( C \cap \{ g > 0 \} \), \( T_n g < \infty \) on \( D \).

**Corollary 1.** Let \( T \) be a sub-Markovian operator, let \( g_i \to g \) as \( i \to \infty \), \( g_i \in L^+_1 \), \( \sup_i g_i \leq g \). Then

\[
(10) \quad \lim_{n \to \infty} \sum_{i=0}^{n-1} T^i g_i
\]

is infinite on the set \( C \cap \{ g > 0 \} \) and is finite on the set \( D \).

**Proof.** The assertion about the behavior on \( C \) follows from Theorem 1 applied with \( A = C, f_{n_i} = f > 0, f \in L_1, g_{n_i} = g_i \). On the other hand, on \( D \) we have

\[
(11) \quad \lim_{n \to \infty} \sum_{i=0}^{n-1} T^i g_i \leq \lim_{n \to \infty} \sum_{i=0}^{n-1} T^i \left( \sup_i g_i \right)
\]

which is finite by Hopf's theorem.

For semi-Markovian operators the corollary remains valid with \( C \) replaced by \( YC^h \), \( D \) replaced by \( YD^h \).

3. A counterexample. Is it possible to replace the assumption (1) in Theorem 1 by the weaker assumption of uniform integrability of sequences? We particularize as follows. Let \( (v_i), (\varphi_i), i = 1, 2, \ldots \), be sequences of positive constants satisfying

\[
(12) \quad \sum_i \varphi_i = 1, \quad v_n \varphi_n \to 0, \quad \sum_n v_n \varphi_n / n = \infty.
\]

Now let \( (X, \alpha, \mu) \) be a probability space such that there is a measurable partition \( \{ A_i \} \) of \( X \) with \( \mu(A_i) = \varphi_i, i = 1, 2, \ldots \). Let \( g_{n_i} = 1 \) for all \( n, i \) and let

\[
f_{n_i} = f_i = v_i \quad \text{on } A_i,
\]

\[
= 0 \quad \text{on } X - A_i.
\]
The sequence \( f_i \) converges to \( f = 0 \). The uniform integrability in the presence of pointwise convergence to zero is equivalent with the convergence of the integrals \( \int f_i \, d\mu = v_i \rho_i \) to \( \int 0 \, d\mu = 0 \); thus \( (f_i) \) is uniformly integrable. Moreover, on \( A_n \)

\[
\sup_n \sum_{i=1}^n f_i/n = \sup_n f_n/n = v_n/n;
\]

hence

\[
\int \left( \sup_n \sum_{i=1}^n f_i/n \right) d\mu = \sum_n v_n \rho_n/n = \infty.
\]

By a theorem of Blackwell and Dubins [1], there is, on a suitable probability space \((X^*, \mathcal{A}^*, \mu^*)\), a sequence of functions \((f_i^*)\) with the same joint distribution as \((f_i)\), and a \(\sigma\)-field \(\mathcal{C}\) such that

\[
\mu^* \left\{ \frac{1}{n} \sum_i^n E(f_i \mid \mathcal{C}) \to 0 \right\} = 0.
\]

We identify the starred and the nonstarred expressions. Let the operator \( T \) be the conditional expectation \( E(\cdot \mid \mathcal{C}) \). Then \( T1 = 1, T^2 = T, T \) satisfies the assumptions \((cA)\) and \((rA)\) with \( A = X \), but by \((15)\) the equality \((2)\) fails on the entire space \( X \).

**References**


**Ohio State University**