

THE AREA AND VARIATION OF LINEARLY CONTINUOUS FUNCTIONS

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1. Introduction. Casper Goffman has developed a theory of area for functions of several variables whose gradients are totally finite vector-valued measures, [G1], [G2], [G3]. In the special case of linearly continuous functions of two variables, he has shown in [G2] that the area is equal to the Hausdorff 2-dimensional measure of the "essential" part of the graph. A similar result is valid for linearly continuous functions of several variables (3.6). In this paper, a co-area formula is established (3.7) which relates the integral of the Hausdorff $n-1$ dimensional measure of the slices by hyperplanes of the essential portion of the graph to the total variation of the gradient measure. In other situations, this has been shown to be a very useful tool, [F3], [GE], [Z1], [Z2].

Our results have also been established independently by Herbert Federer by different methods. His results will be incorporated in his forthcoming book *Geometric measure theory*.

2. Notation and preliminaries. Throughout, $x = (x_1, \dots, x_n)$ will be an arbitrary point of Euclidean n -space E_n . L_n will denote n -dimensional Lebesgue measure and for $k \leq n$, H^k and I^k will stand for k -dimensional Hausdorff measure and k -dimensional integralgeometric measure respectively. In particular, $H^0(A)$ is the number of points in the set A . Recall that H^k and I^k agree on the (H^k, k) rectifiable sets of finite H^k measure, cf. [F2, Theorem 9.7]. G_n is the orthogonal group on E_n and ϕ_n the Haar measure on G_n such that $\phi_n(G_n) = 1$. With $R \in G_n$ and $w \in E_{n-1}$ associate the line $\lambda(R, w)$ given by

$$(1) \quad \lambda(R, w) = R(E_n \cap \{x: x_i = w_i, i = 1, 2, \dots, n-1\}).$$

Under the map λ , the image of the product measure $\phi_n \times L_{n-1}$ is the Haar measure on the space of all lines in E_n .

Let BV stand for the space of all locally summable functions on E_n whose every distribution theory partial derivative is a totally finite measure. For $f \in \text{BV}$, these measures will be denoted by $D_1(f), \dots, D_n(f)$. A function is called *linearly continuous* if, in some

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coordinate system, f is continuous on almost all lines parallel to the coordinate axes. Functions of compact support which are BV and linearly continuous will be the objects of interest in this paper. Denote this class of functions by \mathcal{L} . In [G3], Goffman established the remarkable fact that functions in \mathcal{L} are invariant under bilipschitzian maps. In particular, a function $f \in \mathcal{L}$ is linearly continuous with respect to all rectangular coordinate systems. Thus, we have the following

2.1. THEOREM. *If $f \in \mathcal{L}$ and $R \in G_n$, then f is continuous on $\lambda(R, w)$ for L_{n-1} almost every $w \in E_{n-1}$.*

For $f \in \mathcal{L}$, consider the set of integral means

$$f_k(x) = L_n[S(x, k^{-1})]^{-1} \int_{S(x, k^{-1})} f(y) dL_n(y),$$

where $S(x, k^{-1})$ is the n -ball of radius k^{-1} and center x . Let β_f be the total variation of the gradient measure $(D_1(f), \dots, D_n(f))$ and α_f the total variation of $(L_n, D_1(f), \dots, D_n(f))$. If β_k and α_k are the corresponding measures for f_k , then for every open rectangle $S \subset E_n$, it was shown in [G3, Lemma 1] that

$$(2) \quad \lim_{k \rightarrow \infty} \alpha_k(S) = \alpha_f(S), \quad \lim_{k \rightarrow \infty} \beta_k(S) = \beta_f(S).$$

Also, for $k=1, 2, \dots$, and $i=1, 2, \dots, n$,

$$(3) \quad \beta_k(S_k) \leq \beta_f(S) \quad \text{and} \quad |D_i(f_k)|(S_k) \leq |D_i(f)|(S),$$

where $|D_i(f)|$ denotes the total variation of the measure $D_i(f)$, and where S_k is the set of points in S whose distance from $E_n - S$ is greater than k^{-1} .

3. **Co-area and integralgeometric measure.** Throughout the remainder of the paper, we will consider a function $f \in \mathcal{L}$ defined on the open unit cube Q . Since the integral means f_k are BV and continuous, they are of bounded variation in the sense of Tonelli (BVT) and therefore, the proof of Theorem 3.1 in [Z1] gives

$$(4) \quad \beta_k(Q) = \kappa \int_{G_n} \int_{E_{n-1}} \int_{E_1} H^0[f_k^{-1}(t) \cap \lambda(R, w)] dL_1(t) dL_{n-1}(w) d\phi_n(R),$$

where κ is the normalizing constant used in the definition of $n-1$ dimensional integralgeometric measure [F2, 2.16]. In view of the Banach indicatrix formula for the variation of continuous functions of a real variable, it follows from [G3, p. 169] that

$$| D_n(f) | (Q) = \int_{E_{n-1}} \int_{E_1} H^0[f^{-1}(t) \cap \lambda(R_0, w)] dL_1(t) dL_{n-1}(w),$$

where $R_0: E_n \rightarrow E_n$ is the identity. Thus, from (3) and the coordinate invariance of f , for every $R \in G_n$,

$$\begin{aligned} | D_n(f_k \circ R) | (Q_k) &\leq | D_n(f \circ R) | (Q) \\ (5) \qquad \qquad \qquad &= \int_{E_{n-1}} \int_{E_1} H^0[f^{-1}(t) \cap \lambda(R, w)] dL_1(t) dL_{n-1}(w) \\ &\leq \beta_f(Q). \end{aligned}$$

But $| D_n(f_k \circ R) | (Q_k) \rightarrow | D_n(f \circ R) | (Q)$ for every $R \in G_n$, and thus, Lebesgue's dominated convergence theorem along with (2), (4), and (5) leads to

$$(6) \quad \beta_f(Q) = \kappa \int_{G_n} \int_{E_{n-1}} \int_{E_1} H^0[f^{-1}(t) \cap \lambda(R, w)] dL_1(t) dL_{n-1}(w) d\phi_n(R).$$

A similar result would have been obtained if f had been restricted to an open rectangle of Q . Every open set is the union of nonoverlapping closed rectangles and β_f vanishes on the boundary of every rectangle, [G3, §7]. Also, whether open or closed rectangles are used will not affect the value of the integral in (6). Thus, we have

3.1. LEMMA. *If $f \in \mathcal{L}$ is defined on Q and U is an open subset of Q , then*

$$\beta_f(U) = \kappa \int_{G_n} \int_{E_{n-1}} \int_{E_1} H^0[f^{-1}(t) \cap U \cap \lambda(R, w)] dL_1(t) dL_{n-1}(w) d\phi_n(R).$$

If $B \subset Q$ is a Borel set for which $\beta_f(B) = 0$, then for every $\epsilon > 0$ there is an open set $U \supset B$ such that $\beta_f(U) < \epsilon$. Thus, the preceding lemma leads directly to

3.2. LEMMA. *If B is a Borel set for which $\beta_f(B) = 0$, then*

$$\kappa \int_{G_n} \int_{E_{n-1}} \int_{E_1} H^0[f^{-1}(t) \cap B \cap \lambda(R, w)] dL_1(t) dL_{n-1}(w) d\phi_n(R) = 0.$$

For a set $A \subset Q$, let A^i denote the projection of A into E_{n-1} in the x_i direction. The following fact was very useful in [G3]: for every positive integer k there is a closed set $T_k^i \subset Q^i$ with the property that, if $T_k = \bigcup_{i=1}^n T_k^i \times [0, 1]$, then

- (i) f is continuous on T_k ,
- (7) (ii) $L_{n-1}(Q^i - T_k^i) < k^{-1}$, $i = 1, 2, \dots, n$,
- (iii) $\alpha_f(Q - T_k) < k^{-1}$.

Let

$$(8) \quad E_f = \cup T_k \quad \text{and} \quad Z_f = Q - E_f.$$

Notice that $\alpha_f(Z_f) = \beta_f(Z_f) = L_n(Z_f) = 0$ and that Z_f projects into a set of L_{n-1} measure zero in each coordinate direction. It will be shown that f restricted to E_f describes the essential behavior of f on Q . As a first step in this direction, we see that 3.1 and 3.2 imply

$$(9) \quad \begin{aligned} & \beta_f(Q) \\ &= \kappa \int_{G_n} \int_{E_{n-1}} \int_{E_1} H^0[f^{-1}(t) \cap E_f \cap \lambda(R, w)] dL_1(t) dL_{n-1}(w) d\phi_n(R). \end{aligned}$$

3.3. DEFINITION. With the function $f \in \mathfrak{L}$, associate the function $f^*: Q \rightarrow E_{n+1}$ defined by $f(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n))$. Also, for every set $A \subset Q$, let $A^* = f^*(A)$.

It is clear that $E_f^* = f^*(E_f)$ is an analytic (in fact, Borel) subset of E_{n+1} . Thus, for every $t \in E_1$, $f^{-1}(t) \cap E_f$ is an analytic set of E_n , since it is the projection of the intersection of E_f^* with a hyperplane at height t . Since $f^{-1}(t) \cap E_f$ is an analytic set, [F2, 5.11] gives

$$(10) \quad \begin{aligned} & I^{n-1}[f^{-1}(t) \cap E_f] \\ &= \kappa \int_{G_n} \int_{E_{n-1}} H^0[f^{-1}(t) \cap E_f \cap \lambda(R, w)] dL_{n-1}(w) d\phi_n(R). \end{aligned}$$

Assuming for the moment that the integrand in (9) is $L_1 \times L_{n-1} \times \phi_n$ measurable, (9), (10), and Fubini's theorem yield the following expression for β_f :

$$(11) \quad \beta_f(Q) = \int_{E_1} I^{n-1}[f^{-1}(t) \cap E_f] dL_1(t).$$

To see that the integrand in (9) is $L_1 \times L_{n-1} \times \phi_n$ measurable, consider the projection $P: E_{n+1} \rightarrow E_n$ defined by

$$P(x_1, \dots, x_{n-1}, x_n, x_{n+1}) = (x_1, \dots, x_{n-1}, x_{n+1}).$$

For $R \in G_n$, let $R': E_{n+1} \rightarrow E_{n+1}$ be defined by $R'(x_1, \dots, x_{n+1}) = (x'_1, \dots, x'_n, x_{n+1})$, where $R(x_1, \dots, x_n) = (x'_1, \dots, x'_n)$. Since E_f^* is an analytic set, there is a standard procedure to show that the multiplicity function $N[PR'f^*, E_f, y]$ is $L_n \times G_n$ measurable, cf.

[F2, 2.17]. Here, $N[PR'f^*, E_f, y]$ means the number of points in E_f that are mapped into y by $PR'f^*$. If y is written as (w, t) then $N[PR'f^*, E_f, (w, t)]$ is $L_1 \times L_{n-1} \times \phi_n$ measurable. But

$$(12) \quad N[PR'f^*, E_f, (w, t)] = H^0[f^{-1}(t) \cap E_f \cap \lambda(R, w)]$$

and thus, (11) is established.

Define a Borel measure ν_f by

$$\nu_f(B) = \int_{E_1} I^{n-1}[f^{-1}(t) \cap B \cap E_f] dL_1(t),$$

the L_1 measurability of the integrand being established as above. Then, by the same reasoning used to obtain (11), it is clear that β_f and ν_f agree on all open rectangles and therefore, on all Borel sets. Therefore, we have proved the following theorem.

3.4. THEOREM. *If $f \in \mathcal{L}$ and E_f is the set defined in (8), then*

$$\beta_f(B) = \int_{E_1} I^{n-1}[f^{-1}(t) \cap B \cap E_f] dL_1(t)$$

for every Borel set B .

It follows from Stepanoff's Theorem that a function $f \in \mathcal{L}$ has an approximate total differential L_n almost everywhere and thus, by [F1], there are disjoint Borel sets A_0, A_1, \dots whose union is Q with $L_n(A_0) = 0$ such that f is Lipschitz on A_k , $k = 1, 2, \dots$. Therefore, the following theorem has some interest.

3.5. THEOREM. *If $f \in \mathcal{L}$ is Lipschitz on a Borel set $B \subset Q$, then $H^n(B^*) = \alpha_f(B)$.*

PROOF. For each open rectangle $S \subset Q$, it follows from Proposition 7 in [G1] that

$$\alpha_f(S) \geq \int_S (1 + |\nabla_{f_{ap}}|^2)^{1/2} dL_n.$$

Thus, the same inequality must be valid for all Borel sets. By considering a Lipschitz extension of f on B to all of E_n [RR, p. 341], we have from [F2, 5.9] that

$$\int_B (1 + |\nabla_{f_{ap}}|^2)^{1/2} dL_n = H^n[f^*(B)] = H^n(B^*).$$

Thus,

$$(13) \quad \alpha_f(B) \geq H^n(B^*).$$

To obtain the opposite inequality, let F be the Lipschitz extension of f on B to all of E_n and recall from [F3, 3.1] that

$$(14) \quad \begin{aligned} & \int_{B \cap E_f} |\nabla f_{ap}| dL_n \\ &= \int_{B \cap E_f} |\nabla F| dL_n = \int_{E_1} H^{n-1}[F^{-1}(t) \cap B \cap E_f] dL_1(t) \\ &= \int_{E_1} H^{n-1}[f^{-1}(t) \cap B \cap E_f] dL_1(t). \end{aligned}$$

Since $(B \cap E_f)^* = f^*(B \cap E_f)$ if (H^n, n) is rectifiable, by appealing again to the proof of 3.1 in [Z1], it follows that $f^{-1}(t) \cap B \cap E_f$ is $(H^{n-1}, n-1)$ rectifiable for L_1 almost for all $t \in E_1$. Thus, (3.4) and (14) yield for every Borel set B ,

$$(15) \quad \begin{aligned} \beta_f(B \cap E_f) &= \int_{E_1} H^{n-1}[f^{-1}(t) \cap B \cap E_f] dL_1(t) \\ &= \int_{B \cap E_f} |\nabla f_{ap}| dL_n. \end{aligned}$$

As in [G1, §6], for every $\epsilon > 0$, there is a finite Borel partition $\{B_k\}$ of $B \cap E_f$ such that

$$\sum_k \left[L_n(B_k)^2 + \left(\int_{B_k} |\nabla f_{ap}| \right)^2 \right]^{1/2} < \int_{B \cap E_f} (1 + |\nabla f_{ap}|^2)^{1/2} + \epsilon/2$$

and

$$\alpha_f(B \cap E_f) < \sum_k (L_n(B_k)^2 + \beta_f(B_k)^2)^{1/2} + \epsilon/2.$$

Hence, from (15) and [F2, 5.9],

$$(16) \quad \alpha_f(B \cap E_f) < H^n[(B \cap E_f)^*] + \epsilon.$$

Since ϵ is arbitrary, (13) and (16) imply

$$H^n[(B \cap E_f)^*] \leq H^n(B^*) \leq \alpha_f(B) = \alpha_f(B \cap E_f) \leq H^n[(B \cap E_f)^*],$$

which establishes the lemma.

Let $G = Q \times E_1 \cap \{(p, t) : t \leq f(p)\}$ and let $P(G)$ denote those points at which the exterior normal to G exists [F4]. Define the map $\pi: E_{n+1} \rightarrow E_n$ by $\pi(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n)$. The following result

was obtained independently by the author and Herbert Federer. Since our proof is somewhat similar to Federer's, it will not be given here. Federer's proof will appear in his forthcoming book *Geometric measure theory*.

3.6. THEOREM. *If f is BV , then*

$$\alpha_f(B) = H^n[P(G) \cap \pi^{-1}(B)]$$

for every Borel set $B \subset Q$.

In the case that $f \in \mathcal{L}$, proceeding as in [G3, §6], it is easy to find a Borel set $T \subset Q$ such that $\alpha_f(Q - T) = 0$ and f is approximately continuous at every point of T . Thus, letting $A_f = T \cap \pi[P(G)] \cap E_f$, we have $\alpha_f(Q - A_f) = 0$ and it is simple to verify that $A_f^* \subset P(G)$. $P(G)$ is (H^n, n) rectifiable and thus so is A_f^* . It follows that $f^{-1}(t) \cap A_f$ is $(H^{n-1}, n-1)$ rectifiable for L_1 almost all $t \in E_1$, and therefore the following theorem follows directly from 3.4.

3.7. THEOREM. *If $f \in \mathcal{L}$, then*

$$\beta_f(B) = \int H^{n-1}[f^{-1}(t) \cap B \cap A_f] dL_1(t)$$

for every Borel set $B \subset Q$.

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