EIGENVALUE OF THE SQUARE OF A FUNCTION

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In this note we prove a comparison theorem between the smallest eigenvalue of two related differential equations. Specifically, consider the two eigenvalue problems.

(1) \[ y'' + \lambda p y = 0, \quad y(0) = y(T) = 0, \]
and

(2) \[ y'' + \lambda p^2 y = 0, \quad y(0) = y(T) = 0. \]

Theorem. If the smallest eigenvalue \( \lambda_0 \) of (1) is positive, then the smallest eigenvalue \( \lambda_1 \) of (2) satisfies \( \lambda_1 \leq (T \lambda_0 / \pi)^2 \).

Proof. From Leighton [1, Lemma 1], it follows that

(3) \[ \int_0^T (\lambda_0 p - \lambda_1 p^2) y_0^2 \geq 0, \]

where \( y_0 \) is the first eigenfunction of (1). From Wirtinger's inequality [2, p. 184], applied to \( y_0 (x) - y_0 (0) \), we have

\[ \int_0^T (y_0')^2 \leq (T/\pi)^2 \int_0^T (y_0'')^2 = (T/\pi)^2 \int_0^T \lambda_0 y_0^2. \]

On the other hand, multiplying (1) by \( y_0 \) and integrating by parts yields \( f_0^T (y_0')^2 = f_0^T \lambda_0 p y_0^2 \), and thus

(4) \[ \int_0^T [\lambda_0 p - \lambda_0^2 p^2 / \pi^2 ] y_0^2 \leq 0. \]

Add (3) and (4) to get the result.

If \( p \) is a positive constant, then equality holds. In a certain sense, the above is the best possible for strictly positive \( p \). If \( p \) is bounded away from zero and bounded above, then \( p^{2-n} \to 1 \) uniformly, and consequently the smallest eigenvalue \( \lambda_n \) of problem (1) has limit \( (\pi/T)^2 \). The above theorem gives \( \lambda_{n-1} \leq \lambda_n^2 (T/\pi)^2 \) and equality holds in the limit.

As an example, for \( p = x \), it is known that \( \lambda_0 = 9\alpha_0^2 / 4T^2 \), where \( \alpha_0 \) is the smallest positive zero of \( J_{1/3} \), the Bessel function of order 1/3. The above theorem implies that \( \lambda_1 \leq 81\alpha_0^2 / 16\pi^2 T^4 \). In [3, Theorem 2] it is

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proved that \( T \geq 3\alpha_0/2\lambda_1 T^2 \), hence \( \lambda_1 \geq 9\alpha_0^2/4 T^4 \) \((\alpha_0 = 2.9025 \cdots )\). Thus easily obtainable \textit{a priori} bounds are available for \( \lambda_1 \).

References


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