EIGENVALUE OF THE SQUARE OF A FUNCTION

A. M. FINK

In this note we prove a comparison theorem between the smallest eigenvalue of two related differential equations. Specifically, consider the two eigenvalue problems.

$$y'' + \lambda p y = 0, \qquad y(0) = y(T) = 0, \tag{1}$$

and

$$y'' + \lambda p^2 y = 0, \qquad y(0) = y(T) = 0. \tag{2}$$

THEOREM. *If the smallest eigenvalue* λ_0 *of* (1) *is positive, then the smallest eigenvalue* λ_1 *of* (2) *satisfies* $\lambda_1 \leqq (T\lambda_0/\pi)^2$.

PROOF. From Leighton [1, Lemma 1], it follows that

$$\int_0^T (\lambda_0 p - \lambda_1 p^2) y_0^2 \geqq 0, \tag{3}$$

where y_0 is the first eigenfunction of (1). From Wirtinger's inequality [2, p. 184], applied to $y_0'(x) - y_0'(0)$, we have

$$\int_0^T (y_0')^2 \leqq (T/\pi)^2 \int_0^T (y_0'')^2 = (T/\pi)^2 \int_0^T \lambda_0^2 p^2 y_0^2.$$

On the other hand, multiplying (1) by y_0 and integrating by parts yields $\int_0^T (y_0')^2 = \int_0^T \lambda_0 p y_0^2$, and thus

$$\int_0^T [\lambda_0 p - p^2 \lambda_0^2 T^2/\pi^2] y_0^2 \leqq 0. \tag{4}$$

Add (3) and (4) to get the result.

If p is a positive constant, then equality holds. In a certain sense, the above is the best possible for strictly positive p. If p is bounded away from zero and bounded above, then $p^{2^{-n}} \to 1$ uniformly, and consequently the smallest eigenvalue λ_n of problem (1) has limit $(\pi/T)^2$. The above theorem gives $\lambda_{n-1} \leq \lambda_n^2 (T/\pi)^2$ and equality holds in the limit.

As an example, for $p = x$, it is known that $\lambda_0 = 9\alpha_0^2/4T^3$, where α_0 is the smallest positive zero of $J_{1/3}$, the Bessel function of order $1/3$. The above theorem implies that $\lambda_1 \leq 81\alpha_0^4/16\pi^2 T^4$. In [3, Theorem 2] it is

Received by the editors October 18, 1967.

proved that $T \geqq 3\alpha_0/2\lambda_1 T^2$, hence $\lambda_1 \geqq 9\alpha_0^2/4T^4$ ($\alpha_0 = 2.9025 \cdots$). Thus easily obtainable *a priori* bounds are available for λ_1.

References

1. W. Leighton, *On the zeros of solutions of a second-order linear differential equation*, J. Math. Pures Appl. **44** (1965), 297–310.

2. G. H. Hardy, J. E. Littlewood and G. Polya, *Inequalities*, 2nd ed., Cambridge Univ. Press, 1952.

3. A. M. Fink, *On the zeroes of $y''+py=0$ with linear, convex, and concave p*, J. Math. Pures Appl. **46** (1967), 1–10.

Iowa State University