SOME PROPERTIES OF A GENERALIZED HAUSDORFF MEAN

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1. **Introduction.** If d is a moment sequence, i.e. $d_n = \int_{[0,1]} I^n dg$, $n = 0, 1, 2, \cdots$, it is well known that the Hausdorff mean $H(d)$ is conservative if and only if there is a solution g of the foregoing Hausdorff moment problem which is of bounded variation on [0, 1]. Also well known are necessary and sufficient conditions for the existence of such a mass function g. The discovery by J. H. Wells [**5**] of necessary and sufficient conditions for the existence of a quasicontinuous solution of the Hausdorff moment problem, and the extension of these conditions by the author [**4**] to the case of a solution Riemann-integrable on [0, 1], have motivated the search for a generalization of the Hausdorff transformation which under certain conditions preserves convergence if the associated moment sequence is generated by a Riemann-integrable mass function.

In the classical theory of Hausdorff summability the mean $H(d)$ may be developed from the sequence-to-sequence transformation

$$u_n = \Delta^n t_0 = \sum_{p=0}^{n} (-1)^p \binom{n}{p} t_p.$$

We begin with the transformation

$$u_n = \sum_{p=0}^{n} (-1)^p \binom{n}{p}_s t_p$$

such that if s is a sequence of positive numbers and n, p is a nonnegative integer pair, then

$$\binom{n}{p}_s$$

denotes 0 if $n < p$, 1 if $n = p$, and $s_n \cdot s_{n-1} \cdots s_{p+1}/(n-p)!$ if $n > p$. If S denotes the matrix

$$\left[(-1)^p \binom{n}{p}_s\right],$$

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d a number sequence and D the diagonal matrix $[\delta_{np}d_p]$, and if $H^{(s)} = S^{-1}DS$, then $H^{(s)}$ is the matrix corresponding to the generalized Hausdorff mean which we denote by $H^{(s)}(d)$. It is readily seen that if $s_n = n$, $n = 1, 2, 3, \cdots$, then $H^{(s)}(d)$ is $H(d)$. Parallel with the classical theory we find that $S^{-1} = S$,

$$H_{np}^{(s)} = \binom{n}{p}_s \Delta^{n-p} d_p,$$

if b is a number sequence, the matrices corresponding to $H^{(s)}(d)$ and $H^{(s)}(b)$ commute, and if X is a matrix commutable with the matrix of $H^{(s)}(d)$, then, provided $d_n = d_m$ implies $n = m$, there is a sequence b such that X is the matrix of $H^{(s)}(b)$. Furthermore, this generalization is essentially the only one with the above properties.

With g restricted to a function of bounded variation on $[0, 1]$,[1] conditions for convergence-preservation by $H^{(s)}(d)$ are established, an example of a nonconservative $H^{(s)}(d)$ mean is given, and the convergence domains of $H(d)$ and $H^{(s)}(d)$ are compared.

2. **The generalized mean.** The motivation for the generalization chosen is to retain the form

$$u_n = \sum_{p=0}^{n} \binom{n}{p} \Delta^{n-p} d_p t_p$$

of the Hausdorff transformation in the case $s_n = n$, $n = 1, 2, 3, \cdots$.

THEOREM 1. *$S^{-1} = S$ and for each nonnegative integer pair n, p,*

$$H_{np}^{(s)} = \binom{n}{p}_s \Delta^{n-p} d_p.$$

PROOF. From the definitions, $S_{nn}^2 = 1$ and $S_{np} = 0$ if $n < p$. If $n > p$, then

$$\begin{aligned} \sum_{k=0}^{n} S_{nk} S_{kp} &= \sum_{k=p}^{n} (-1)^k \binom{n}{k}_s (-1)^p \binom{k}{p}_s \\ &= \sum_{k=p}^{n} (-1)^{k+p} s_n \cdot s_{n-1} \cdots s_{p+1}/(n-k)!(k-p)! \\ &= \sum_{k=p}^{n} (-1)^{k+p} \binom{n-p}{n-k} \binom{n}{p}_s = 0. \end{aligned}$$

[1] The author wishes to thank the referee for suggesting this restriction in the present paper as well as for his other very helpful comments.

Hence $S^2=I$.

If n, p is a nonnegative integer pair, then

$$\begin{aligned} H_{np}^{(s)} &= \sum_{k=0}^{n} S_{nk}d_kS_{kp} \\ &= \sum_{k=p}^{n} (-1)^{k+p}\binom{n}{p}_s\binom{n-p}{n-k}d_k \\ &= \binom{n}{p}_s \sum_{k=0}^{n-p}(-1)^k\binom{n-p}{n-p-k}d_{p+k} \\ &= \binom{n}{p}_s \Delta^{n-p}d_p. \end{aligned}$$

We next observe that the $H^{(s)}(d)$ mean is essentially unique with respect to the desired properties.

THEOREM 2. *Suppose* (i) *G is a row-finite matrix of numbers such that $G^{-1}=G$, $G_{00}=1$, and if $n\geqq 1$, $G_{n0}>0$ and $G_{n,n-1}\neq 0$.*

(ii) *D is a diagonal matrix of numbers.*

(iii) *$A=GDG$.*

(iv) *For each nonnegative integer pair n,p with $n\geqq p$, $A_{np}=|G_{np}|\Delta^{n-p}d_p$.*

Then if $s_n=|G_{n,n-1}|$, $n=1, 2, 3, \cdots$, $G=S$.

PROOF. A simple computation shows that $G_{nn}=(-1)^n$ and $G_{20}=-G_{10}G_{21}/2>0$. Suppose k is the least positive integer such that there is a positive integer r, $r<k$, such that

$$G_{kr}\neq(-1)^r\binom{k}{r}_s.$$

Now

$$A_{k0}=\sum_{p=0}^{k}G_{kp}d_pG_{p0}=G_{k0}\Delta^kd_0=G_{k0}\sum_{p=0}^{k}(-1)^p\binom{k}{p}d_p.$$

Since $G_{k0}>0$,

$$G_{kr}\binom{r}{0}_s=G_{k0}(-1)^r\binom{k}{r},\qquad 0<r<k,$$

or

$$G_{k0} = |G_{k1}| s_1 / \binom{k}{1} = |G_{k2}| s_2 s_1/2! \binom{k}{2} = \cdots$$

$$= |G_{k,k-2}| s_{k-2}s_{k-3} \cdots s_1/(k-2)! \binom{k}{k-2}$$

$$= |G_{k,k-1}| s_{k-1}s_{k-2} \cdots s_1/(k-1)! \binom{k}{k-1}.$$

But $|G_{k,k-1}| = s_k$, whence $|G_{k,k-2}| = s_k s_{k-1}/2!$, and it follows by induction that

$$|G_{kr}| = \binom{k}{r}_s, \qquad r = 0, 1, \cdots, k,$$

and

$$G_{kr} = (-1)^r \binom{k}{r}_s.$$

The next two theorems are stated without proofs as they may be established by methods analogous to those used in [**3**, Theorems 197, 198, p. 249].

THEOREM 3. *If each of b and d is a number sequence,* $B = [\delta_{np} b_p]$, $X = SBS$ *and* $H^{(s)} = SDS$, *then* $H^{(s)}X = XH^{(s)}$.

THEOREM 4. *If d is a number sequence such that* $d_n = d_m$ *implies* $n = m$, $H^{(s)} = SDS$ *and X is a matrix such that* $H^{(s)}X = XH^{(s)}$, *then there is a number sequence b such that X is the matrix of* $H^{(s)}(b)$.

3. **Convergence-preservation.** Before considering more general moment sequences it seems advisable to investigate the $H^{(s)}(d)$ method in the context of the $H(d)$ method. Accordingly the moment sequences in this paper will be restricted to those generated by a function of bounded variation on $[0, 1]$, and we denote by BV the space of such sequences. Furthermore, the treatment will be considerably simplified by restricting s so that $s_n \leqq n$, $n = 1, 2, 3, \cdots$.

We observe that if $0 < \alpha < 1$ and $s_n = n - 1 + \alpha$, then $H^{(s)}(d)$ is essentially the $H^{(\alpha)}(\mu)$ method studied extensively by K. Endl [**2**] who, however, does not place this restriction on α.

If $d \in BV$, it is apparent that with $s_n \leqq n$, $n \geqq 1$, an $H^{(s)}(d)$ mean satisfies two of the Silverman-Toeplitz conditions for convergence-preservation, so that we need consider only the sequence of the row-sums of $H^{(s)}$. If $p \geqq 0$, let $\pi^{(p)}$ denote the sequence such that $\pi_p^{(p)} = 1$

and $\pi_n^{(p)}=\prod_{k=p+1}^n s_k/k$, $n>p$. With π replacing $\pi^{(0)}$ and ρ denoting the sequence $\{1/\pi_n\}$, we have the following lemma, the routine proof of which is omitted.

LEMMA. *If* $n\geqq 0$,

$$\sum_{p=0}^{n} H_{np}^{(s)} = \sum_{p=0}^{n} \pi_n^{(p)} H_{np} = \pi_n \sum_{p=0}^{n} H_{np}\rho_p = \pi_n \sum_{p=0}^{n} (-1)^p \binom{n}{p} \Delta^p \rho_0 d_p$$

and

$$\sum_{p=0}^{n} H_{np} = \rho_n \sum_{p=0}^{n} H_{np}^{(s)} \pi_p.$$

There is a fundamental theorem which follows directly from the lemma.

THEOREM 5. *If* $d\in BV$, *an* $H^{(s)}(d)$ *mean is conservative if and only if the matrix*

$$\left[(-1)^p \pi_n \binom{n}{p} \Delta^p \rho_0\right]$$

is conservative over the space BV.

The next three theorems give sufficient conditions for convergence-preservation. To facilitate comparison of the conditions, we introduce the sequence a such that $a_n=1-s_n/n$, $n=1, 2, 3, \cdots$.

THEOREM 6. *If* $d\in BV$ *and* $\sum_n a_n$ *is convergent, then* $H^{(s)}(d)$ *is conservative.*

PROOF. If x is a convergent sequence and $l_0=\lim_n H_{n0}$, then $\lim_H x =\chi(H)\lim x+l_0x_0$ where $\chi(H)=d_0-l_0$ [6, p. 93]. If $u=\lim \pi$, then $u>0$, so that $\lim \rho=1/u$. Therefore, using the lemma,

$$\lim_n \sum_{p=0}^{n} H_{np}^{(s)} = u \lim_n \sum_{p=0}^{n} H_{np}\rho_p = u[(d_0 - l_0)/u + l_0] = d_0 + l_0(u - 1).$$

COROLLARY. *If* $d\in BV$ *and* $\sum_n a_n$ *is convergent, then* $H^{(s)}(d)$ *is regular if and only if* $H(d)$ *is regular.*

THEOREM 7. *If* $d\in BV$ *and for each positive integer* n, $a_{n+1}\geqq na_n/(n+1)$, *then* $H^{(s)}(d)$ *is multiplicative.*

PROOF. Since d is the difference of totally monotone sequences, it may be assumed that d is totally monotone. From the lemma,

$$\sum_{p=0}^{n} H_{np}^{(s)} = \pi_n \sum_{p=0}^{n} H_{np}\rho_p$$

$$= \pi_n \sum_{p=0}^{n} \rho_p \binom{n}{p}\left[H_{n+1,p} \Big/ \binom{n+1}{p} + H_{n+1,p+1} \Big/ \binom{n+1}{p+1}\right]$$

$$= \pi_n H_{n+1,0} + \pi_n \sum_{p=1}^{n} H_{n+1,p}\left[\binom{n}{p-1}\rho_{p-1} + \binom{n}{p}\rho_p\right] \Big/ \binom{n+1}{p}$$

$$+ d_{n+1}$$

$$= \pi_n H_{n+1,0} + \pi_n \sum_{p=1}^{n} H_{n+1,p}[p\rho_{p-1} + (n - p + 1)\rho_p]/(n + 1)$$

$$+ d_{n+1}.$$

If $1 \leqq p \leqq n$, then

$$\pi_n H_{n+1,p}[p\rho_{p-1} + (n - p + 1)\rho_p]/(n + 1) - H_{n+1,p}^{(s)}$$

$$= \pi_n H_{n+1,p}\rho_p\{[p\rho_{p-1}\pi_p + (n - p + 1)]/(n + 1) - s_{n+1}/(n + 1)\}$$

$$= \pi_n H_{n+1,p}\rho_p\{[p(1 - a_p) + (n - p + 1)]/(n + 1) - (1 - a_{n+1})\}$$

$$= \pi_n H_{n+1,p}\rho_p\{a_{n+1} - pa_p/(n + 1)\} \geqq 0.$$

Therefore,

$$\sum_{p=0}^{n+1} H_{n+1,p}^{(s)} = H_{n+1,0}^{(s)} + \sum_{p=1}^{n+1} H_{n+1,p}^{(s)}$$

$$\leqq \sum_{p=0}^{n} H_{np}^{(s)} - (\pi_n - \pi_{n+1})H_{n+1,0}$$

$$\leqq \sum_{p=0}^{n} H_{np}^{(s)}$$

since $H_{n+1,0} \geqq 0$ and $\pi_{n+1} \leqq \pi_n$. Furthermore, $\sum_n a_n$ is divergent so that $\lim \pi = 0$.

It is interesting to note that if $0 < \alpha < 1$ and $s_n = n - 1 + \alpha$, $n \geqq 1$, then $a_n = (1 - \alpha)/n$, and the $H^{(\alpha)}(\mu)$ method of Endl [2, pp. 426–429] satisfies the condition of the theorem with $a_{n+1} = na_n/(n+1)$, $n = 1, 2, 3, \cdots$.

THEOREM 8. *If $d \in BV$ and for each positive integer n, $a_{n+1} \leqq na_n/(n+1)$, then $H^{(s)}(d)$ is conservative.*

PROOF. In view of Theorem 6 we may assume $\sum_n a_n$ to be divergent. Following the argument for Theorem 7 yields

$$\sum_{p=0}^{n+1} H_{n+1,p}^{(s)} \geqq \sum_{p=0}^{n} H_{np}^{(s)} - (\pi_n - \pi_{n+1})H_{n+1,0}.$$

Hence if k is a positive integer,

$$\sum_{p=0}^{n+k} H_{n+k,p}^{(s)} \geqq \sum_{p=0}^{n} H_{np}^{(s)} - (\pi_n - \pi_{n+k})d_0.$$

Since $\lim \pi = 0$, if $\epsilon > 0$, there is a positive integer N_1 such that if $n > N_1$, then

$$\sum_{p=0}^{n+k} H_{n+k,p}^{(s)} > \sum_{p=0}^{n} H_{np}^{(s)} - \epsilon.$$

Since

$$\sum_{p=0}^{n} H_{np}^{(s)} \leqq d_0, \qquad n \geqq 0,$$

there is a positive integer N, $N \geqq N_1$, such that if $n > N$, then

$$\sum_{p=0}^{n+k} H_{n+k,p}^{(s)} < \sum_{p=0}^{n} H_{np}^{(s)} + \epsilon.$$

For an example of a nonconservative $H^{(s)}(d)$ mean we take $H(d)$ to be the Cesàro mean of order one and let $a_n = 3/4$ if $n = 2^{2k-1}$, $k = 1, 2, 3, \cdots$, with $a_n = 0$ otherwise. Then if $n = 2^{2k-1} - 1$, $\sum_{p=0}^{n} H_{np}^{(s)} \geqq 3/4$, while if $n = 2^{2k-1}$, $\sum_{p=0}^{n} H_{np}^{(s)} \leqq 1/2$.

4. The inclusion problem. With m denoting the space of bounded sequences, we consider the efficiency over m of $H(d)$ and $H^{(s)}(d)$.

THEOREM 9. *If $d \in BV$ and* $\lim \pi > 0$, *then $H^{(s)}(d)$ and $H(d)$ are equivalent over n.*

PROOF. Suppose $x \in m$ and $v = \lim_H x$. Then if it exists,

$$\lim_{H^{(s)}} x = \lim_n \pi_n \sum_{p=0}^{n} H_{np}\rho_p x_p = \lim \pi \lim_n \sum_{p=0}^{n} H_{np} x_p \rho_p.$$

If $u = \lim \pi$ and $l_0 = \lim_n H_{n0}$, then [6, p. 93]

$$\begin{aligned}\lim_{H^{(s)}} x &= u\left[\left(\lim_H x - \sum_p \lim_n H_{np}x_p\right)\lim \rho + \sum_p \rho_p \lim_n H_{np}x_p\right] \\ &= u[(v - l_0x_0)/u + \rho_0 l_0 x_0] \\ &= v - l_0x_0(1-u).\end{aligned}$$

Suppose $w=\lim_{H^{(s)}}x$. Then

$$\begin{aligned}\lim_H x &= \lim \rho \lim_n \sum_{p=0}^{n} H_{np}^{(s)} x_p \pi_p \\ &= (1/u)[(w - ul_0x_0)u + \pi_0 ul_0x_0] \\ &= w - l_0x_0(u-1).\end{aligned}$$

Since from the definition of $H^{(s)}(d)$ it is apparent that $[H^{(s)}]^{-1}=SD^{-1}S$, we may, for the case $\lim d\neq 0$, compare the relative strengths of an $H^{(s)}(d)$ mean and an $H(d)$ mean by using conditions on the inverse transformations.

THEOREM 10. *If $d\in BV$ and $H^{(s)}(d)$ and $H^{-1}(d)$ are conservative, then $H^{(s)}(d)$ and $H(d)$ are equivalent.*

PROOF. If x is a divergent sequence and Hx is convergent, then $H^{-1}(Hx)$ is convergent. Hence $H(d)$ sums no divergent sequence. Since $\|H^{-1}\|$ exists, $\|[H^{(s)}]^{-1}\|$ exists, and it is well known (e.g., see [**6**, p. 232]) that $[H^{(s)}]^{-1}(d)$ is conservative, so that $H^{(s)}(d)$ sums no divergent sequence.

The foregoing argument also establishes the following theorem.

THEOREM 11. *If $d\in BV$ and $H^{(s)}(d)$ and $[H^{(s)}]^{-1}(d)$ are conservative, then $H(d)$ includes $H^{(s)}(d)$.*

Finally, the case in which $H^{(s)}(d)$ is conservative and neither $H^{-1}(d)$ nor $[H^{(s)}]^{-1}(d)$ is conservative remains an open question, some aspects of which are dealt with in the following comments.

Suppose $d\in BV$, $\lim d=0$ and $H(d)$ is regular. Then there is a bounded sequence which is not $H(d)$-summable. If $s_n/n=1/2^n$, $n=1, 2, 3, \cdots$, then $\lim_n \sum_{p=0}^{n} |H_{np}^{(s)}|=0$, so that $H^{(s)}(d)$ sums every bounded sequence and thus includes $H(d)$. Let σ denote the set of divergent sequences such that $t\in\sigma$ if and only if $t_i=0$ or 1, $i=0, 1, 2, \cdots$. If $H(d)$ is a Cesàro mean of order r, $r\geqq 1$, then almost all sequences of σ are $H(d)$-summable to 1/2 [**1**, pp. 211–212]. If $\sum_n a_n$ is divergent and x is a convergent sequence, there is a number w such that $\lim_{H^{(s)}} x=wx$, and it follows from [**1**, p. 211, 8.6, IV] that almost all sequences of σ are $H^{(s)}(d)$-summable to $w/2$. On the other hand, if $\lim d\neq 0$ and $H(d)$ is multiplicative, then neither $H(d)$ nor $H^{(s)}(d)$ sums to the same limit each of the sequences of a subset of σ which is of positive measure.

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