CONVERGENCE-PRESERVATION CRITERIA FOR A GENERALIZED HAUSDORFF MEAN

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1. **Introduction.** The generalized Hausdorff method denoted by $H^{(s)}(d)$ is defined in [**4**] to the effect that if

(1) s is a sequence of positive numbers,

(2) $d_n = \int_{[0,1]} I^n dg$, $n = 0, 1, 2, \cdots$, and

(3) where n, p is a nonnegative integer pair,

$$\binom{n}{p}_s$$

denotes 0 if $n < p$, 1 if $n = p$, and $s_n \cdot s_{n-1} \cdots s_{p+1}/(n-p)!$ if $n > p$, then

$$H_{np}^{(s)} = \binom{n}{p}_s \Delta^{n-p} d_p.$$

If $s_n = n$, $n = 1, 2, 3, \cdots$, then $H^{(s)}$ is denoted by H and $H(d)$ represents the Hausdorff method. Furthermore $H^{(s)}$ is expressible in terms of H by the formula $H_{np}^{(s)} = \pi_n^{(p)} H_{np}$, where $\pi_p^{(p)} = 1$ and $\pi_n^{(p)} = \prod_{k=p+1}^{n} s_k\, k$, $n > p$. As in [**4**] s is restricted so that $s_n \leqq n$, $n = 1, 2, 3, \cdots$.

Let R denote the space of sequences d each of which is generated by a function g which is Riemann-integrable on $[0, 1]$, and let BV denote the subspace of R such that $d \in BV$ if g is of bounded variation on $[0, 1]$. $H^{(s)}(d)$ means generated by sequences of the latter type were considered in [**4**]. In this paper we are primarily interested in moment sequences belonging to R but not to BV. It is found that the three conditions of Silverman and Toeplitz are reduced to two, and a generalization is obtained of one of the fundamental theorems of $H(d)$-summability. Conditions on s and g under which $H^{(s)}(d)$ is multiplicative are established and an example is indicated to show that the restriction on g is not necessary.

2. **Modification of the Silverman-Toeplitz conditions.** If $d \in R \backslash BV$, the three conditions for convergence-preservation reduce to two in the case of an $H^{(s)}(d)$ mean, namely, that $\| H^{(s)} \|$ exists and $\{ \sum_{p=0}^{n} H_{np}^{(s)} \}$ converges. Before establishing this we have, with c_0 denoting the space of zero-limit sequences, the following property of the collection of product sequences.

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LEMMA 1. *If there is a nonnegative integer P such that $\pi^{(P)} \in c_0$, then for each nonnegative integer p, $\pi^{(p)} \in c_0$.*

PROOF. If $p < P$ and $n \geqq 1$, $\pi_n^{(p)} \leqq \pi_n^{(P)}$. If $\epsilon > 0$ and $p > P$, then there is a positive integer N such that if $n > N$, $\pi_n^{(P)} < \epsilon \prod_{k=P+1}^{p} s_k/k$, so that $\pi_n^{(p)} = \pi_n^{(P)} / \prod_{k=P+1}^{p} s_k/k < \epsilon$.

We next show that if $d \in R \backslash BV$ and $H^{(s)}(d)$ is conservative, then it is multiplicative.

THEOREM 1. *If $d \in R\backslash BV$ and there is a number K such that $\sum_{p=0}^{n} |H_{np}^{(s)}| < K$, $n = 0, 1, 2, \cdots$, then $\lim_n H_{np}^{(s)} = 0$, $p = 0, 1, 2, \cdots$.*

PROOF. Since $d \notin BV$, the sequence $\{ \sum_{p=0}^{n} |H_{np}| \}$ is unbounded. Suppose there is a nonnegative integer P such that $\pi^{(P)} \notin c_0$. From Lemma 1 if $p \geqq 0$, there is a positive number t_p such that $\lim \pi^{(p)} = t_p$. Furthermore $t_0 \leqq t_p$ so that if $n \geqq p$, $\pi_n^{(p)} \geqq \pi_n^{(0)} \geqq t_0$ and $\sum_{p=0}^{n} |H_{np}^{(s)}| \geqq t_0 \sum_{p=0}^{n} |H_{np}|$. Therefore $\pi^{(p)} \in c_0$ if $p \geqq 0$, and since the set $\{H_{np}\}$ is bounded, $\lim_n H_{np}^{(s)} = 0$.

It may be observed that if $d \in R\backslash BV$, then the existence of $\|H^{(s)}\|$ is necessary and sufficient that $H^{(s)}(d)$ be regular over the space c_0 [**2**, p. 49, Theorem 4].

There remains only the statement of conditions necessary and sufficient for convergence-preservation.

THEOREM 2. *If $d \in R\backslash BV$, then $H^{(s)}(d)$ is multiplicative if and only if there exist numbers K and L such that*

(i) $\sum_{p=0}^{n} |H_{np}^{(s)}| < K$, $n = 0, 1, 2, \cdots$;

(ii) $\lim_n \sum_{p=0}^{n} H_{np}^{(s)} = L$.

3. **A fundamental theorem.** If $a_n = 1 - s_n/n$, $n = 1, 2, 3, \cdots$, it is apparent from the proof of Theorem 1 that if $d \in R\backslash BV$ and $H^{(s)}(d)$ is conservative, then $\sum_n a_n$ is divergent. Hence in view of [**4**, Theorem 6] we may extend one of the fundamental theorems of Hausdorff summability.

THEOREM 3. *If $\sum_n a_n$ is convergent, then $H^{(s)}(d)$ is conservative if and only if $d \in BV$.*

4. **Sufficient conditions for convergence-preservation.** For each nonnegative integer pair n, p, $n \geqq p$, let f_{np} denote the polynomial

$$\binom{n}{p} I^p (1 - I)^{n-p}$$

and $V_{[a,b]}f_{np}$ the variation of f_{np} on $[a, b]$. Then $H_{np}=\int_{[0,1]}f_{np}dg$. We state some lemmas for convenience.

LEMMA 2. *If n, p is a positive integer pair, $n>p$, then*
(i) *f_{np} is increasing on $[0, p/n]$ and decreasing on $[p/n, 1]$;*
(ii) $V_{[0,1]}f_{np}=2f_{np}(p/n)$;
(iii) $V_{[0,1]}f_{n,n-p}=V_{[0,1]}f_{np}$;
(iv) $\lim_n V_{[0,1]}f_{np}=2p^pe^{-p}/p!$.

LEMMA 3. $\lim_p p^pe^{-p}/p!=0$.

LEMMA 4. *If $d\in R$ and $\epsilon>0$, there is a positive integer pair N, P such that if $n>N$ and $P\leqq p\leqq n-P$, then $|H_{np}|<\epsilon$.*

LEMMA 5. *If $0<t<1$ and p is a positive integer, then* $\lim_n V_{[0,t]}f_{n,n-p}=0$.

Lemma 2 may be established by computing f'_{np} and noting that the high point of f_{np} is at p/n, and Lemma 3 follows from an application of Stirling's formula. Lemma 5 is readily obtained from Lemma 2. Since the author is not aware of the existence in the literature of a proof of Lemma 4, an argument therefor is given after the theorem.

THEOREM 4. *If $d\in R\backslash BV$ and there is a number M such that $\sum_{p=0}^{n}\pi_n^{(p)}<M$, then $H^{(s)}(d)$ is regular over c_0. Furthermore if $g(1-)$ exists, $H^{(s)}(d)$ is multiplicative, and if $g(1)-g(1-)=2$, $H^{(s)}(d)$ is regular.*

PROOF. If W is a number such that $|H_{np}|<W$, $n\geqq p$, $p=0, 1, 2, \cdots$, then $\sum_{p=0}^{n}|H_{np}^{(s)}|=\sum_{p=0}^{n}\pi_n^{(p)}|H_{np}|<MW$, and $H^{(s)}(d)$ is regular over c_0.

If $\epsilon>0$, from Lemma 4 there is a positive integer pair N_1, P such that if $n>N_1$, $\sum_{p=P}^{n-P}|H_{np}^{(s)}|=\sum_{p=P}^{n-P}\pi_n^{(p)}|H_{np}|<\epsilon/3$.

From Theorem 1 there is a positive integer N_2, $N_2\geqq N_1$, such that if $n>N_2$, then $\sum_{p=0}^{P-1}|H_{np}^{(s)}|<\epsilon/3$.

From Lemma 2(iv) there is a number V such that if $0<p<P$ and $n>p$, then $V_{[0,1]}f_{np}<V$. Hence by Lemma 2(iii), if $n-P<n-p<n$, then $V_{[0,1]}f_{n,n-p}<V$. If $\epsilon>0$, there is a positive number t such that if $t\leqq x<1$, then $|g(x)-g(1-)|<\epsilon/9PV$. Let U denote an upper bound on $[0, 1]$ for $|g|$. From Lemma 5 there is a positive integer N, $N\geqq N_2$, such that if $n>N$ and $0<p<P$, then $V_{[0,t]}f_{n,n-p}<\epsilon/18PU$. Since from Lemma 2(i) $f_{n,n-p}$ is decreasing on $[(n-p)/n, 1]$, for each n there is a number z_n, $t<z_n<1$, such that $V_{[z_n,1]}f_{n,n-p}<\epsilon/18PU$. Thus if $n>N$,

$$|H^{(s)}_{n,n-p}| \leqq \left|\int_{[0,1]} f_{n,n-p}dg\right| \leqq \left|\int_{[0,t]}[g-g(1-)]df_{n,n-p}\right|$$
$$+\left|\int_{[z_n,1]}[g-g(1-)]df_{n,n-p}\right|+\left|\int_{[t,z_n]}[g-g(1-)]df_{n,n-p}\right|$$
$$< 2UV_{[0,t]}f_{n,n-p}+(\epsilon/9PV)V_{[t,z_n]}f_{n,n-p}+2UV_{[z_n,1]}f_{n,n-p}<\epsilon/3P.$$

Therefore if $n>N$, then $\sum_{p=0}^{n-1}|H^{(s)}_{np}|<\epsilon$, and it is well known (e.g., see [3]) that $\lim d=[g(1)-g(1-)]/2$.

An example of a function g showing that the existence of $g(1-)$ is not necessary for convergence-preservation may be constructed by defining $g(x)=h(1-x)$ where h is defined in [1, p. 119, Theorem 3] and observing that $f_{n,n-p}(x)=f_{np}(1-x)$, $0\leqq x\leqq 1$.

5. **Proof of Lemma 4.** For the convenience of the reader the lemma is restated.

LEMMA 4. *If $d\in R$ and $\epsilon>0$, there is a positive integer pair N, P such that if $n>N$ and $P\leqq p\leqq n-P$, then $|H_{np}|<\epsilon$.*

PROOF. If $\epsilon>0$, then from Lemmas 2(iv) and 3 there is a positive integer pair N_P, P such that if $n>N_P$, then $V_{[0,1]}f_{nP}<\epsilon/U$, where $|g|<U$ on $[0, 1]$.

We next show that if n, p is a positive integer pair, $p\leqq n/2$, and $p\leqq q\leqq n-p$, then $V_{[0,1]}f_{nq}\leqq V_{[0,1]}f_{np}$. If $p=n/2$, then $q=p$. If $n=3$, then $q=p$ or $q=n-p$, and the conclusion follows from Lemma 2(iii). Suppose there is a least positive integer, denoted by $k+1$, such that if $p<(k+1)/2$, then there is a positive integer q such that $p\leqq q\leqq k+1-p$ and $V_{[0,1]}f_{k+1,q}>V_{[0,1]}f_{k+1,p}$. From Lemma 2(iii) $q<k+1-p$. Furthermore if $p\leqq q\leqq k-p$, $V_{[0,1]}f_{kq}\leqq V_{[0,1]}f_{kp}$. Since the sequence $\{(1+1/n)^n\}$ is increasing, $(k/[k+1])^k(1+1/[k-q])^{k-q}V_{[0,1]}f_{kq}\leqq(k/[k+1])^k(1+1/[k-p])^{k-p}V_{[0,1]}f_{kp}$, whence using Lemma 2(ii), $V_{[0,1]}f_{k+1,q}\leqq V_{[0,1]}f_{k+1,p}$.

Thus if $N=N_P+2P$ and $n>N$, then $|\int_{[0,1]}f_{np}dg|=|\int_{[0,1]}gdf_{np}|<\epsilon$.

REFERENCES

1. J. P. Brannen, *Concerning Hausdorff matrices and absolutely convergent sequences,* Proc. Amer. Math. Soc. **15** (1964), 114–123.

2. G. H. Hardy, *Divergent series,* Clarendon Press, Oxford, 1956.

3. R. E. Lane, *The integral of a function with respect to a function,* Proc. Amer. Math. Soc. **5** (1954), 69–76.

4. C. W. Leininger, *Some properties of a generalized Hausdorff mean,* Proc. Amer. Math. Soc. **20** (1969), 88–96.

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