1. Introduction. The generalized Hausdorff method denoted by
\( H^{(s)}(d) \) is defined in [4] to the effect that if
(1) \( s \) is a sequence of positive numbers,
(2) \( d_n = \int_{0}^{1} I^s dg, \ n = 0, 1, 2, \ldots \), and
(3) where \( n, \ p \) is a nonnegative integer pair,
\[
\begin{pmatrix} n \\ p \end{pmatrix}_s
\]
denotes 0 if \( n < p \), 1 if \( n = p \), and \( s_n \cdot s_{n-1} \cdots s_{p+1} / (n - p)! \) if \( n > p \), then
\[
H^{(s)}_{np} = \begin{pmatrix} n \\ p \end{pmatrix}_s \Delta^{n-p} d_p.
\]
If \( s_n = n \), \( n = 1, 2, 3, \ldots \), then \( H^{(s)} \) is denoted by \( H \) and \( H(d) \)
represents the Hausdorff method. Furthermore \( H^{(s)} \) is expressible
in terms of \( H \) by the formula \( H^{(s)}_{np} = \pi^{(p)}_n H_{np} \), where \( \pi^{(p)}_n = 1 \) and
\[
\pi^{(p)}_n = \prod_{k=1}^{p+1} \pi^{(p+1)}_{n+k} k, \ n > p.
\]
As in [4] \( s \) is restricted so that \( s_n \leq n, \ n = 1, 2, 3, \ldots \).

Let \( R \) denote the space of sequences \( d \) each of which is generated
by a function \( g \) which is Riemann-integrable on \( [0, 1] \), and let \( B V \)
denote the subspace of \( R \) such that \( d \in B V \) if \( g \) is of bounded variation
on \( [0, 1] \). \( H^{(s)}(d) \) means generated by sequences of the latter type
were considered in [4]. In this paper we are primarily interested in
moment sequences belonging to \( R \) but not to \( B V \). It is found that the
three conditions of Silverman and Toeplitz are reduced to two, and a
generalization is obtained of one of the fundamental theorems of
\( H(d) \)-summability. Conditions on \( s \) and \( g \) under which \( H^{(s)}(d) \) is
multiplicative are established and an example is indicated to show
that the restriction on \( g \) is not necessary.

2. Modification of the Silverman-Toeplitz conditions. If \( d \in R \setminus B V \),
the three conditions for convergence-preservation reduce to two in
the case of an \( H^{(s)}(d) \) mean, namely, that \( \| H^{(s)} \| \) exists and \( \{ \sum_{p=0}^{n} H^{(s)}_{np} \} \)
converges. Before establishing this we have, with \( c_0 \) denoting the
space of zero-limit sequences, the following property of the collection
of product sequences.

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Lemma 1. If there is a nonnegative integer $P$ such that $\pi^{(P)} \subseteq c_0$, then for each nonnegative integer $p$, $\pi^{(p)} \subseteq c_0$.

Proof. If $p < P$ and $n \geq 1$, $\pi^{(p)}_n \leq \pi^{(P)}_n$. If $\epsilon > 0$ and $p > P$, then there is a positive integer $N$ such that if $n > N$, $\pi^{(P)}_n < \epsilon \prod_{k=p+1}^n s_k/k$, so that $\pi^{(p)}_n = \pi^{(P)}_n / \prod_{k=p+1}^n s_k/k < \epsilon$.

We next show that if $d \in R \setminus BV$ and $H^{(\sigma)}(d)$ is conservative, then it is multiplicative.

Theorem 1. If $d \in R \setminus BV$ and there is a number $K$ such that $\sum_{n=0}^\infty |H^{(\sigma)}_{np}| < K$, $n = 0, 1, 2, \ldots$, then $\lim_{n \to \infty} H^{(\sigma)}_{np} = 0$, $p = 0, 1, 2, \ldots$.

Proof. Since $d \notin BV$, the sequence $\{ \sum_{n=0}^\infty |H^{(\sigma)}_{np}| \}$ is unbounded. Suppose there is a nonnegative integer $P$ such that $\pi^{(P)} \subseteq c_0$. From Lemma 1 if $p \geq 0$, there is a positive number $t_p$ such that $\lim_{n \to \infty} \pi^{(P)} = t_p$. Furthermore $t_0 \leq t_p$ so that if $n \geq p$, $\pi^{(P)}_n \leq \pi^{(P)}_n \leq t_0$ and $\sum_{n=0}^\infty |H^{(\sigma)}_{np}| \leq t_0 \sum_{n=0}^\infty |H^{(\sigma)}_{np}|$. Therefore $\pi^{(P)} \subseteq c_0$ if $p \geq 0$, and since the set $\{ H^{(\sigma)}_{np} \}$ is bounded, $\lim_n H^{(\sigma)}_{np} = 0$.

It may be observed that if $d \in R \setminus BV$, then the existence of $\|H^{(\sigma)}\|$ is necessary and sufficient that $H^{(\sigma)}(d)$ be regular over the space $c_0$ [2, p. 49, Theorem 4].

There remains only the statement of conditions necessary and sufficient for convergence-preservation.

Theorem 2. If $d \in R \setminus BV$, then $H^{(\sigma)}(d)$ is multiplicative if and only if there exist numbers $K$ and $L$ such that

(i) $\sum_{n=0}^\infty |H^{(\sigma)}_{np}| < K$, $n = 0, 1, 2, \ldots$;
(ii) $\lim_{n \to \infty} \sum_{p=0}^n H^{(\sigma)}_{np} = L$.

3. A fundamental theorem. If $a_n = 1 - s_n/n$, $n = 1, 2, 3, \ldots$, it is apparent from the proof of Theorem 1 that if $d \in R \setminus BV$ and $H^{(\sigma)}(d)$ is conservative, then $\sum_n a_n$ is divergent. Hence in view of [4, Theorem 6] we may extend one of the fundamental theorems of Hausdorff summability.

Theorem 3. If $\sum_n a_n$ is convergent, then $H^{(\sigma)}(d)$ is conservative if and only if $d \in BV$.

4. Sufficient conditions for convergence-preservation. For each nonnegative integer pair $n, p$, $n \geq p$, let $f_{np}$ denote the polynomial

$${n \choose p} P(1 - I)^{n-p}$$
and \( V_{[a,b]} f_{np} \) the variation of \( f_{np} \) on \([a, b]\). Then \( H_{np} = \int_{[0,1]} f_{np} dg \). We state some lemmas for convenience.

**Lemma 2.** If \( n, p \) is a positive integer pair, \( n > p \), then

1. \( f_{np} \) is increasing on \([0, p/n]\) and decreasing on \([p/n, 1]\);
2. \( V_{[0,1]} f_{np} = 2f_{np}(p/n) \);
3. \( V_{[0,1]} f_{n,n-p} = V_{[0,1]} f_{np} \);
4. \( \lim_n V_{[0,1]} f_{np} = 2p^pe^{-p}/p! \).

**Lemma 3.** \( \lim_p p^pe^{-p}/p! = 0 \).

**Lemma 4.** If \( d \in \mathbb{R} \) and \( \epsilon > 0 \), there is a positive integer pair \( N, P \) such that if \( n > N \) and \( P \leq p \leq n - P \), then \( |H_{np}| < \epsilon \).

**Lemma 5.** If \( 0 < t < 1 \) and \( p \) is a positive integer, then \( \lim_n V_{[0,t]} f_{n,n-p} = 0 \).

Lemma 2 may be established by computing \( f_{np}' \) and noting that the high point of \( f_{np} \) is at \( p/n \), and Lemma 3 follows from an application of Stirling's formula. Lemma 5 is readily obtained from Lemma 2. Since the author is not aware of the existence in the literature of a proof of Lemma 4, an argument therefor is given after the theorem.

**Theorem 4.** If \( d \in \mathbb{R} \setminus BV \) and there is a number \( M \) such that \( \sum_{n=0}^{n} |p_n| < M \), then \( H^{(e)}(d) \) is regular over \( c_0 \). Furthermore if \( g(1-) \exists \), \( H^{(e)}(d) \) is multiplicative, and if \( g(1) - g(1-) = 2 \), \( H^{(e)}(d) \) is regular.

**Proof.** If \( W \) is a number such that \( |H_{np}| < W \), \( n \geq p \), \( p = 0, 1, 2, \ldots \), then \( \sum_{n=0}^{n} |H_{np}^{(e)}| = \sum_{n=0}^{n} p_n |H_{np}| < MW \), and \( H^{(e)}(d) \) is regular over \( c_0 \).

If \( \epsilon > 0 \), from Lemma 4 there is a positive integer pair \( N_1, P \) such that if \( n > N_1 \), \( \sum_{n=0}^{n-P} |H_{np}^{(e)}| = \sum_{n=0}^{n-P} p_n |H_{np}| < \epsilon/3 \).

From Theorem 1 there is a positive integer \( N_2 \), \( N_2 \geq N_1 \), such that if \( n > N_2 \), then \( \sum_{n=0}^{n-P} |H_{np}^{(e)}| < \epsilon/3 \).

From Lemma 2(iv) there is a number \( V \) such that if \( 0 < p < P \) and \( n > P \), then \( V_{[0,1]} f_{np} < V \). Hence by Lemma 2(iii), if \( n-P < n-p < n \), then \( V_{[0,1]} f_{n,n-p} < V \). If \( \epsilon > 0 \), there is a positive number \( t \) such that if \( t < \epsilon < 1 \), then \( |g(x) - g(1-)| < \epsilon/9PV \). Let \( U \) denote an upper bound on \([0, 1]\) for \(|g|\). From Lemma 5 there is a positive integer \( N \), \( N \geq N_2 \), such that if \( n > N \) and \( 0 < p < P \), then \( V_{[0,n]} f_{n,n-p} < \epsilon/18PU \). Since from Lemma 2(i) \( f_{n,n-p} \) is decreasing on \([(n-p)/n, 1]\), for each \( n \) there is a number \( z_n, t < z_n < 1 \), such that \( V_{[z_n,1]} f_{n,n-p} < \epsilon/18PU \). Thus if \( n > N \),
\[ |H_{n,n-p}^{(n)}| \leq \left| \int_{[0,1]} f_{n,n-p} dg \right| \leq \left| \int_{[0,1]} [g - g(1-)] df_{n,n-p} \right| \]

\[ + \left| \int_{[t,n]} [g - g(1-)] df_{n,n-p} \right| + \left| \int_{[t,n]} [g - g(1-)] df_{n,n-p} \right| \]

\[ < 2UV_{[0,t]} f_{n,n-p} + (e/9PV) V_{[t,n]} f_{n,n-p} + 2UV_{[t,n]} f_{n,n-p} < \epsilon/3P. \]

Therefore if \( n > N \), then \( \sum_{n=1}^{\infty} |H_{np}^{(n)}| < \epsilon \), and it is well known (e.g., see [3]) that \( \lim d = [g(1) - g(1-)]/2. \)

An example of a function \( g \) showing that the existence of \( g(1-) \) is not necessary for convergence-preservation may be constructed by defining \( g(x) = h(x) \), where \( h \) is defined in [1, p. 119, Theorem 3] and observing that \( B_{d}(x) = B_{d}(1-x) \), \( 0 \leq x \leq 1. \)

5. Proof of Lemma 4. For the convenience of the reader the lemma is restated.

**Lemma 4.** If \( d \in R \) and \( \epsilon > 0 \), there is a positive integer pair \( N, P \) such that if \( n > N \) and \( P \leq p \leq n - P \), then \( |H_{np}| < \epsilon. \)

**Proof.** If \( \epsilon > 0 \), then from Lemmas 2(iv) and 3 there is a positive integer pair \( N_p, P \) such that if \( n > N_p \), then \( V_{[0,1]} f_{n,p} < \epsilon/U \), where \( |g| < c \) on \([0,1]\).

We next show that if \( n, p \) is a positive integer pair, \( p \leq n/2 \), and \( P \leq q \leq n - p \), then \( V_{[0,1]} f_{nq} \leq V_{[0,1]} f_{np} \). If \( p = n/2 \), then \( q = p \). If \( n = 3 \), then \( q = p \) or \( q = n - p \), and the conclusion follows from Lemma 2(iii). Suppose there is a least positive integer, denoted by \( k + 1 \), such that if \( p < (k + 1)/2 \), then there is a positive integer \( q \) such that \( p \leq q \leq k + 1 - p \) and \( V_{[0,1]} f_{k+1,p} > V_{[0,1]} f_{k+1,p} \). From Lemma 2(iii) \( q < k + 1 - p \). Furthermore if \( p \leq q \leq k - p \), \( V_{[0,1]} f_{kq} \leq V_{[0,1]} f_{kp} \). Since the sequence \( \{(1+1/n)^n\} \) is increasing, \( (k/[k+1])^m \leq (1+1/\gamma) \leq V_{[0,1]} f_{kq} \leq (k/[k+1])^m \leq V_{[0,1]} f_{kp} \), whence using Lemma 2(ii), \( V_{[0,1]} f_{k+1,p} \leq V_{[0,1]} f_{k+1,p} \).

Thus if \( N = N_p + 2P \) and \( n > N \), then \( \left| \int_{[0,1]} f_{n,p} dg \right| = \left| \int_{[0,1]} g df_{n,p} \right| < \epsilon. \)

**References**