COMPARISON THEOREMS AND INTEGRAL INEQUALITIES FOR VOLTERRA INTEGRAL EQUATIONS

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1. **Introduction.** Although comparison theorems for ordinary differential equations (see, for example [1, p. 26], [2, p. 274]) are widely known and used, there appear to be no results of this kind for integral equations. In §3 of this paper we give two comparison theorems for Volterra integral equations of the second kind:

$$(1.1) \qquad y_i(x) = f_i(x) + \int_a^x K(x, t) y_i(t) dt \qquad (i = 1, 2).$$

In the real case, these theorems give sufficient conditions for the validity of the inequality $y_1(x) \leqq y_2(x)$, where y_i is the unique solution of (1.1).

These theorems are simple consequences of well-known facts concerning integral equations of the form (1.1). These facts are briefly summarized in §2; the reader is referred to [3] and [4] for the details. As in [3], [4] we deal with Lebesgue square integrable functions throughout so that all equalities (and inequalities) are of the almost everywhere kind. In this section, we also give an upper bound for $|y_i(x)|$ in (1.1).

The last section of this paper deals with integral inequalities related to Volterra integral equations of the second kind. These results are closely related to the positivity of the operator $L(y) = y - Ky$, and include an extension to the L^2 case of the recent theorem of Chu and Metcalfe [5]. As pointed out in [5] this theorem includes the classical Gronwall inequality and some of its linear extensions. It should also be noted that it was pointed out by Bellman in [6] that such inequalities are closely related to the positivity of operators.

2. **Volterra integral equations.** For completeness, and because the following results are essential in the sequel, we state here the principal results we need for Volterra integral equations.

Let $I = [a, b]$, where $-\infty \leqq a < b \leqq +\infty$. By a *Volterra type kernel on* $I \times I$ we mean any complex-valued function $K \in L^2(I \times I)$ such that $K(x, t) = 0$ for $a \leqq x < t \leqq b$. By the *resolvent kernel of* K *for the (complex) value* λ we mean the function Γ given by the series

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$$(2.1)\qquad \Gamma(x, t; \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} K^{(n)}(x, t),$$

where $K^{(1)} = K$, and for $n \geqq 2$,

$$(2.2)\quad K^{(n)}(x, t) = \int_I K(x, s) K^{(n-1)}(s, t) ds = \int_t^x K(x, s) K^{(n-1)}(s, t) ds.$$

Each of the *iterated kernels* $K^{(n)}$ is also a Volterra type kernel on $I \times I$. The above series is, for each complex λ, mean square convergent on $I \times I$, and Γ is, for almost all $(x, t) \in I \times I$, an entire function of λ. Moreover, Γ is also a Volterra type kernel on $I \times I$.

In what follows we shall always take $\lambda = 1$, and write $\Gamma(x, t; 1) = \Gamma(x, t)$. For each complex-valued function $f \in L^2(I)$, the Volterra integral equation

$$(2.3)\qquad y(x) = f(x) + \int_a^x K(x, t) y(t) dt, \qquad x \in I,$$

has a unique L^2-solution y given by the formula

$$(2.4)\qquad y(x) = f(x) + \int_a^x \Gamma(x, t) f(t) dt, \qquad x \in I.$$

Whenever the function K_1 defined by

$$(2.5)\qquad K_1(x) = \sup_{a \leqq t \leqq x} |K(x, t)|, \qquad x \in I,$$

is integrable over I, we may apply the method used by Bellman in [7] to obtain a bound for $|y(x)|$, as given in

THEOREM 1. *Let K be a Volterra type kernel on $I \times I$, and let $f \in L^2(I)$. If the function K_1 defined by* (2.5) *is integrable over I, and if y is the unique L^2-solution of* (2.3), *then*

$$(2.6)\qquad \begin{aligned} |y(x)| \leqq |f(x)| &+ K_1(x) \exp\left(\int_a^x K_1(s) ds\right) \\ &\cdot \int_a^x |f(t)| \exp\left(-\int_a^t K_1(s) ds\right) dt, \qquad \text{a.e. on } I. \end{aligned}$$

To prove this, note that from (2.3) we have

$$|y(x)| \leqq |f(x)| + K_1(x) \int_a^x |y(t)| dt.$$

Hence, setting $R(x) = \int_a^x |y(t)| dt$, we obtain

$$R'(x) - K_1(x)R(x) \leqq |f(x)|.$$

The rest of the proof follows more or less as in [7]. (See also theorem 5 below.)

We note without proof that by using the *resolvent equation* satisfied by Γ, it follows in the same way that

$$\begin{aligned} |\Gamma(x, t)| \leqq |K(x, t)| + K_1(x, t) \exp\left(\int_t^x K_1(s, t)ds\right) \\ \cdot \int_t^x |K(u, t)| \exp\left(-\int_t^u K_1(s, t)ds\right)du \end{aligned} \tag{2.7}$$

almost everywhere on $I \times I$, provided

$$K_1(x, t) = \max_{t \leqq s \leqq x} |K(x, s,)| \qquad (x, t) \in I \times I$$

is integrable over I for almost all $t \in I$.

3. Positive kernels. Comparison theorems. We begin this section with a simple lemma which, in the case that f and K are nonnegative, gives an obvious lower bound for the unique L^2-solution of equation (2.4).

LEMMA 1. *Let $f \in L^2(I)$ with $f \geqq 0$ on I, and let K be a nonnegative Volterra type kernel on $I \times I$. If Γ is the resolvent kernel of K (for the value $\lambda = 1$), and if y is the unique L^2-solution of* (2.4), *then*

$$\Gamma(x, t) \geqq K(x, t) \qquad \text{a.e. on } I \times I, \tag{3.1}$$

$$y(x) \geqq f(x) \qquad \text{a.e. on } I. \tag{3.2}$$

PROOF. It follows at once from (2.2) that all the iterated kernels of K are nonnegative on $I \times I$. Setting $\lambda = 1$ in (2.1), we obtain (3.1). The inequality (3.2) follows from (3.1) and the representation (2.4) of y since $f \geqq 0$.

We now apply Lemma 1 to prove our first comparison theorem for Volterra integral equations.

THEOREM 2. *Let $f_i \in L^2(I)$, and let K_i, $i = 1, 2$, be Volterra type kernels on $I \times I$, satisfying*

$$|f_1(x)| \leqq f_2(x) \ \text{a.e. on } I, \qquad |K_1(x, t)| \leqq K_2(x, t) \ \text{a.e. on } I \times I. \tag{3.3}$$

If y_i is the unique L^2-solution of the integral equation

$$y_i(x) = f_i(x) + \int_a^x K_i(x, t)y_i(t)dt, \tag{3.4}$$

then $|y_1(x)| \leqq y_2(x)$ *a.e. on* I. *In fact,*

$$y_2(x) - |y_1(x)| \geqq f_2(x) - |f_1(x)| \qquad \text{a.e. on } I. \tag{3.5}$$

PROOF. Note that f_2 and K_2 are *real*-valued, nonnegative functions, whereas f_1, K_1 could be complex-valued. By Lemma 1 it follows that $y_2(x) \geqq f_2(x) \geqq 0$. Since

$$y_2(x) = f_2(x) + \int_a^x K_2(x, t) y_2(t) dt,$$

$$|y_1(x)| \leqq |f_1(x)| + \int_a^x |K_1(x, t)| |y_1(t)| dt$$

$$\leqq |f_1(x)| + \int_a^x K_2(x, t) |y_1(t)| dt,$$

it follows that

$$y_2(x) - |y_1(x)| \geqq \{f_2(x) - |f_1(x)|\} + \int_a^x K_2(x, t)\{y_2(t) - |y_1(t)|\} dt.$$

Letting g denote the (positive) difference between the left side and the right side of this inequality, we have $g \in L^2(I)$, and

$$y_2(x) - |y_1(x)| = g(x) + \{f_2(x) - |f_1(x)|\} + \int_a^x K_2(x, t)\{y_2(t) - |y_1(t)|\} dt.$$

Since $g + (f_2 - |f_1|) \in L^2(I)$, it follows from (3.3) and Lemma 1, that

$$y_2(x) - |y_1(x)| \geqq g(x) + \{f_2(x) - |f_1(x)|\},$$

proving (3.5).

From now on we shall deal only with real-valued functions. In this case the following comparison theorem is sometimes applicable when theorem 2 is not.

THEOREM 3. *Let* $f_i \in L^2(I)$, *and let* K_i, $i = 1, 2$, *be Volterra type kernels on* $I \times I$, *satisfying*

$$\begin{aligned} &0 \leqq f_2(x), \qquad f_1(x) \leqq f_2(x) \quad \text{a.e. on } I, \\ &0 \leqq K_1(x, t) \leqq K_2(x, t) \quad \text{a.e. on } I \times I. \end{aligned} \tag{3.6}$$

If y_i *is the unique* L^2-*solution of the integral equation* (3.4), *then*

$$y_2(x) - y_1(x) \geqq f_2(x) - f_1(x) \qquad \text{a.e. on } I. \tag{3.7}$$

PROOF. Let Γ_i denote the resolvent kernel of K_i, $i=1, 2$. Precisely as in Lemma 1, it follows from the third of the inequalities (3.6) that

$$0 \leqq \Gamma_1(x, t) \leqq \Gamma_2(x, t) \qquad \text{a.e. on } I \times I. \tag{3.8}$$

Using an obvious operator notation, we have $y_i = f_i + \Gamma_i f_i$, hence

$$\begin{aligned} y_2 - y_1 &= (f_2 - f_1) + \Gamma_2 f_2 - \Gamma_1 f_1 \\ &= (f_2 - f_1) + (\Gamma_2 - \Gamma_1) f_2 + \Gamma_1 (f_2 - f_1). \end{aligned}$$

The inequality (3.7) now follows from (3.8) and the first two of the inequalities (3.6).

COROLLARY. *Under the hypotheses* (3.6), (3.7) *can be improved to*

$$y_2 - y_1 \geqq (f_2 - f_1) + K_1(f_2 - f_1) \qquad \text{a.e. on } I. \tag{3.9}$$

This follows from the last equality of the theorem together with (3.1) of Lemma 1.

In the same way, we observe that the conclusions of Lemma 1 and of Theorem 2 can be somewhat improved.

REMARKS. 1. Retaining the hypothesis $K \geqq 0$, we see that if $f \leqq 0$ on I, then the conclusion (3.2) of Lemma 1 becomes $y \leqq f$, or better, $y \leqq f + Kf$. Similarly, if the inequalities in the first two of (3.6) are reversed, then so also are the inequalities (3.7), (3.9).

2. Since any initial value problem for linear differential equations can be reduced to a Volterra integral equation of the second kind [**4**, p. 18] the above results can be used to obtain comparison theorems for such problems.

4. **Positivity of Volterra operators.**

If K is a Volterra type kernel on $I \times I$, we define the linear operator L on $L^2(I)$ by

$$L(u) = u - Ku. \tag{4.1}$$

Following Beckenbach and Bellman [**6**, p. 131], the Volterra operator L is said to be *positive* if $L(u) \geqq 0$ implies that $u \geqq 0$. Lemma 1 provides an immediate sufficient condition for the positivity of L.

THEOREM 4. *The operator L defined by* (4.1) *is positive if* $K(x, t) \geqq 0$ *a.e. on* $I \times I$.

For, if $L(u) = f \geqq 0$, then $f \in L^2(I)$, and $u = f + Ku$. Hence $u \geqq f \geqq 0$ by Lemma 1.

Similarly, according to Remark 1, L is *negative* if K is nonpositive on $I \times I$.

As pointed out in [6], the inequalities of Bellman and Gronwall are closely related to the question of the positivity of operators. The following result which includes these inequalities is a consequence of Theorem 4, and is a slight extension of the theorem in [5].

THEOREM 5. *Let* $f \in L^2(I)$, *and let* K *be a nonnegative Volterra kernel on* $I \times I$. *If* $u \in L^2(I)$, *and*

$$u(x) \leqq f(x) + \int_a^x K(x, t)u(t)dt \qquad a.e. \text{ on } I, \tag{4.2}$$

then $u(x) \leqq y(x)$ *a.e. on* I, *where* y *is the unique* L^2-*solution of the Volterra integral equation* $y = f + Ky$. (*Similarly, if the inequality is reversed in* (4.2), *then* $u(x) \geqq y(x)$ *a.e. on* I.)

PROOF. Set $u - Ku = g$, so $u = g + Ku$ with $g \in L^2(I)$ and $g \leqq f$. Then $L(y-u) = L(y) - L(u) = f - g \geqq 0$ and the conclusion follows from Theorem 4.

We conclude this paper with a few remarks concerning possible converses of Theorem 5. We may, equivalently, formulate such questions in terms of the operator L defined by (4.1). In general we may ask what conditions on K and u will guarantee that $L(u) \geqq 0$? If we consider only nonnegative Volterra kernels K, it is clear that the condition $u \geqq 0$ is *not* a sufficient condition for $L(u) \geqq 0$, although it is necessary by Theorem 4. The simplest sufficient conditions for $L(u) \geqq 0$ appear to be

$$u \geqq 0, \qquad u \text{ nondecreasing on } I, \qquad K \geqq 0, \qquad \int_I K(x, t)dt \leqq 1; \tag{4.3}$$

these conditions are quite restrictive, however.

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