RELATIVE INTERIORS OF CONVEX HULLS

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The purpose of this paper is to review several recent generalizations of a classic theorem of E. Steinitz, to show how they are related, and to prove an extension of the Bonnice-Klee theorem which both generalizes and unifies these results. The theorem of Steinitz [6] is

THEOREM A. *If* $A \subset R^n$ *and* $w \in \text{int conv } A$, *then* $w \in \text{int conv } B$ *for some subset* $B \subset A$ *with* $\text{card } B \leqq 2n$.

Generalizations of this theorem have either tried to characterize when an upper bound of $2n$, $2n-1$, etc. for card B is necessarily assumed (see [4], [5]), have added further conditions on the set A in order to obtain better bounds on card B (see [1]–[4]), or have asked for the bounds on card B if we demand only $w \in \text{int}_d \text{ conv } B$ where $0 \leqq d \leqq n$. (Definition: $w \in \text{int}_d X$ if there is a d-simplex contained in X with w in its relative interior.) The following two results are of the latter two types and are due respectively to Bonnice-Klee [1] and Ives [3].

THEOREM B. *If* $A \subset R^n$ *and* $w \in \text{int}_d \text{ conv } A$ *then* $w \in \text{int}_d \text{ conv } B$ *for some subset* $B \subset A$ *with* $\text{card } B \leqq \max (n+1, 2d)$.

THEOREM C. *If* $A \subset R^n$ *and* $w \in \text{int conv } A$, *let* B *be a subset of* A *of least cardinality such that* $w \in \text{int conv } B$ *and let* k *be the dimension of the highest dimensional simplex with vertices in* A *and having* w *in its relative interior. Then*

$$n + 1 + \{n/k\} \leqq \text{card } B \leqq 2n - k + 1.$$

($\{x\}$ will always denote the largest integer strictly less than x.)

The extension we shall prove is

THEOREM D. *If* $A \subset R^n$ *and* $w \in \text{int}_d \text{ conv } A$, $(0 \leqq d \leqq n)$, *let* B *be a subset of* A *of least cardinality such that* $w \in \text{int}_d \text{ conv } B$ *and let* k *be (as in Theorem* C*) the dimension of the highest dimensional simplex with*

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vertices in A and having w in its relative interior. If $d \leqq k$, then

$$d + 1 \leqq \text{card } B \leqq k + 1$$

and if $k < d$, then

$$d + 1 + \{d/k\} \leqq \text{card } B \leqq \max(2d - k + 1, \min(2k + 2, n + 2)).$$

Furthermore these bounds are as sharp as possible under the given conditions.

Since the proof of Theorem D depends upon Theorem C, we will give an independent proof of that result. The upper bound on card B in Theorem C was established independently in [**2**]. The proof of the lower bound on card B in Theorem C as reported by Ives [**3**] was based on an induction on k. The proof below of the lower bound is straightforward and is an easy consequence of the following useful result in the theory of positive bases [**4**, Theorem 2.6, p. 11]. Also see [**5**].

LEMMA E. *Let B be a positive basis for R^n. Then B admits a partitioning $B = V_1 \cup \cdots \cup V_r$ into pairwise disjoint subsets* $(1 \leqq r \leqq n)$, *such that*

(1) card $V_1 \geqq$ card $V_2 \geqq \cdots \geqq$ card $V_r \geqq 2$.

(2) *For each j, $1 \leqq j \leqq r$ the set* pos $(V_1 \cup \cdots \cup V_j)$ *is a linear subspace of R^n of dimension* $\sum_{i=1}^{j}$ (card $V_i - 1$).

PROOF OF THEOREM C. Let p denote card B. We show only that $n+1+\{n/k\} \leqq p$. Without loss of generality assume w is the origin. Then p has the equivalent definition

$$p = \min\{\text{card } B \mid B \subset A, B \text{ is a positive basis for } R^n\}.$$

Since this minimum is assumed for some set B, let us now assume that B is a positive basis for R^n, $B \subset A$, card $B = p$, and there is no smaller subset of A which forms a positive basis for R^n. Let $B = V_1 \cup \cdots \cup V_r$ be the partitioning of Lemma E.

Since pos V_1 is a linear space of dimension (card $V_1 - 1$), it follows that card $V_1 \leqq k+1$ by the definition of k. Since $n = \sum_{i=1}^{r}(\text{card } V_i - 1) = \text{card } B - r$ is fixed, we see that $p = \text{card } B = n+r$ is a minimum when r is as small as possible. It is clear that this occurs when each pos V_i is a linear subspace of maximal dimension and R^n is their direct linear sum; that is, card $V_1 =$ card $V_2 = \cdots =$ card $V_{r-1} = k+1$, and card $V_r - 1$ is either $k+1$ or the remainder after dividing n by k. Then $r = \{n/k\} + 1$. Thus $p = n + r = n + \{n/k\} + 1$ as was to be shown.

PROOF OF THEOREM D.

Case 1: ($d \leqq k$). If $w \in \text{int}_d$ conv B, then clearly card $B \geqq d+1$. If A is a set of exactly $k+d+2$ points, if $k+d \leqq n$, and if $A = A_1 \cup A_2$ where card $A_1 = k+1$, card $A_2 = d+1$, and the k-dimensional flat aff A_1 meets the d-dimensional flat aff A_2 only at the point $\{w\}$, then $d+1 = \text{card } B$.

Clearly card $B \leqq k+1$, and if A is the set of vertices of a k-simplex which has w in its relative interior then card $B = k+1$. Thus these bounds are the best possible.

Case 2: ($k < d$). Without loss of generality assume $w = 0$, the origin, let $p = \text{card } B$, and let $B = \{b_i\}_{i=1}^{p}$. Because $0 \in \text{int}_d$ conv B there are scalars $\mu_i > 0$ ($\mu_i \neq 0$ by the minimality of B) with $\sum_{i=1}^{p} \mu_i = 1$ such that $0 = \sum_{i=1}^{p} \mu_i b_i$. Thus $-b_j = \sum_{i \neq j} (\mu_i/\mu_j) b_i \in \text{pos } B$ for $j = 1, 2, \cdots, p$. Hence pos B is closed under scalar multiplication and therefore is a linear subspace, say of dimension m. Then $0 \in \text{int}_m$ conv B, and $m \geqq d$. To apply Theorem C in pos B, we note that for no proper subset C of B is $0 \in \text{int}_m$ conv C ($\subset \text{int}_d$ conv C) and that if j is the dimension of the highest dimensional simplex with vertices in B and having 0 in its relative interior, then $j \leqq k$. Thus by Theorem C, card $B \geqq m+1+\{m/j\} \geqq d+1+\{d/k\}$.

To establish the asserted upper bound for card B, let $S \subset A$ be the set of vertices of a k-simplex with 0 in its relative interior. Thus pos S is a k-dimensional subspace of R^n. Extend S to a positive basis $C \subset A$ of R^n.

Keeping in mind the maximality of k, an examination of the proof of Lemma E [**5**], [**4**, Theorem 2.6, p. 11] shows that S can be used as V_1 in the factorization $C = V_1 \cup \cdots \cup V_r$ as given by Lemma E. Thus $k+1 = \text{card } V_1 \geqq \cdots \geqq \text{card } V_r \geqq 2$ and for each j, $1 \leqq j \leqq r$, the set pos $(V_1 \cup \cdots \cup V_j)$ is a linear subspace of R^n of dimension $\sum_{i=1}^{j} (\text{card } V_i - 1)$. Let j be the least integer such that $\sum_{i=1}^{j} (\text{card } V_i - 1) \geqq d$, and let $B' = \bigcup_{i=1}^{j} V_i$. Then $0 \in \text{int}_d$ conv B' and, by the minimality of B, card $B \leqq$ card B'. Thus, to complete the proof, it suffices to show that

$$\text{card } B' \leqq \max (2d - k + 1, \min (2k+2, n+2)).$$

If $j = 2$, then $k+1 = \text{card } V_1 \geqq \text{card } V_2$ implies that card B' $= \text{card } V_1 + \text{card } V_2 \leqq 2k+2$. But also,

$$n \geqq \dim \text{pos } B' = (\text{card } V_1 - 1) + (\text{card } V_2 - 1) = \text{card } B' - 2$$

so card $B' \leqq n+2$. Therefore, if $j = 2$, card $B' \leqq \min (2k+2, n+2)$.

Now suppose that $j \geqq 3$. From $d \geqq 1 + \sum_{i=1}^{j-1} (\text{card } V_i - 1)$, it follows that $2d \geqq -2j + 4 + 2\sum_{i=1}^{j-1} \text{card } V_i$, and so

$$2d - k + 1 \geqq -2j - k + 5 + \text{card } V_1 + \sum_{i=2}^{j-1} \text{card } V_i + \sum_{i=1}^{j-1} \text{card } V_i$$

$$= -2j + 6 + \text{card } V_2 + \sum_{i=3}^{j-1} \text{card } V_i + \sum_{i=1}^{j-1} \text{card } V_i$$

(since card $V_1 = k + 1$)

$$\geqq -2j + 6 + \text{card } V_2 + (j-3)2 + \sum_{i=1}^{j-1} \text{card } V_i$$

(since always card $V_i \geqq 2$)

$$\geqq \text{card } V_j + \sum_{i=1}^{j-1} \text{card } V_i$$

$$= \text{card } B'.$$

Combining the above results for $j=2$ and $j\geqq 3$, we have that always card $B' \leqq \max(2d-k+1, \min(2k+2, n+2))$.

EXAMPLES. Here we show the bounds of Theorem D are the best possible under the given conditions for the case $k<d$. In each example let R^n be the direct linear sum of r linear subspaces pos A_i, $i=1, \cdots, r$ where $k+1=\text{card } A_1 \geqq \text{card } A_2 \geqq \cdots \geqq \text{card } A_r \geqq 2$, and each A_i is the vertex set of a simplex with 0 in its relative interior. In each example, let the given set A of Theorem D be $A = A_1 \cup \cdots \cup A_r$ and let $w=0$.

The lower bound. For any given value of d (where $d>k$) let

$$\text{card } A_1 = \text{card } A_2 = \cdots = \text{card } A_{j-1} = k + 1 \geqq \text{card } A_j \geqq 2$$
$$= \text{card } A_{j+1} = \cdots = \text{card } A_r$$

where j is determined by $d=\sum_{i=1}^{j} (\text{card } A_i - 1)$. Then set B of Theorem D will have cardinality card $B=\sum_{i=1}^{j} \text{card } A_i = d+j = d+\{d/k\}+1$.

The upper bounds. (1) If $r=2$ and $d=k+1$ then card $B=k+1+\text{card } V_2=n+2=\max(2d-k+1, \min(2k+2, n+2))$.

(2) If $r\geqq 2$, and $d=k+1=\text{card } A_i$ for $i=1, \cdots, r$, then $n=rk$ and card $B=2k+2=\max(2d-k+1, \min(2k+2, n+2))$.

(3) If $r=3$, $(k+1)/2 \leqq \text{card } A_2 = \text{card } A_3$ and $d=k+\text{card } A_2$ then

$$\text{card } B = k + 1 + 2 \text{ card } A_2 = 2d - k + 1$$
$$= \max(2d - k + 1, \min(2k + 2, n + 2)).$$

(4) If $r\geqq 2$, and card $A_2 = \cdots = \text{card } A_r = 2$, then

$$\begin{aligned}\text{card } B &= (k+1) + 2(d-k) = 2d - k + 1 \\ &= \max(2d - k + 1, \min(2k+2, n+2)).\end{aligned}$$

Finally, we note that Theorem 2.9 of [1] also gives upper bounds on card B under the hypothesis of Theorem D of this paper but example (2) above and Theorem D show that the conclusion as stated there is incorrect and should read "For $e+1 \leqq d \leqq n$ there is a $Y_d \subset X$ such that pos Y_d is a linear subspace of dimension $\geqq d$ and card $Y_d \leqq 2d-e+1 < 2d$ or card $Y_d \leqq n+2$."

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