MINIMAL SOLUTIONS TO A CLASS OF PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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1. **Introduction.** In [2], the existence of a solution to a generalized optimization problem was considered in which the governing equations were a class of ordinary differential equations. Here a similar argument is used to obtain an existence theorem when the governing equations are a class of parabolic partial differential equations.

2. **Statement of the problem.** Let T be a closed interval $[t^0, t^1]$ of the real line. Suppose Ω is an open set in n-dimensional Euclidean space E^n and let Ω_0 be a compact subset of Ω.

Let $\hat{\mathcal{U}}$ be the set of real-valued functions $u(t, x)$ defined on $T\times\Omega$ and absolutely continuous on T for almost all x such that for some fixed integer k, the function u and all partial derivatives of u with respect to x of order less than or equal to k are in $\mathcal{L}^2(\Omega)$ for fixed t and in $\mathcal{L}^2(T)$ for fixed x.

For each $t\in T$, define the k-Dirichlet norm[2] of u to be

$$\|u\|_k^2 = \sum_{|\alpha|\leq k} \int_{\Omega_0} |D_\alpha u(x, t)|^2 \tag{2.1}$$

where, as is customary, $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)$, the α_i are nonnegative integers and

$$|\alpha| = \alpha_1+\alpha_2+\cdots+\alpha_n.$$

Here D_α denotes the derivative operator

$$D_\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}. \tag{2.2}$$

Let $\mathcal{U}$ be a subset of $\hat{\mathcal{U}}$ with the following additional properties:

(i) For each $u(t, x)$ in $\mathcal{U}$, the partial derivative $u_t(t, x)$ is in $\mathcal{L}^2(T)$.

(ii) The set $\mathcal{U}$ is closed in the sense that if $\{u_n\}$ is a Cauchy sequence in the Dirichlet norm for each t, then there is a function u_0 in $\mathcal{U}$ to which the sequence $\{u_n\}$ converges for each t.

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[2] Strictly speaking, $\|u\|_k$ is not a norm, since it depends upon t, but for convenience we refer to it as such.

Let $\mathcal{V}$ be the set of functions $v(x)$ in $\mathcal{C}_0^\infty(\Omega_0)$, the class of C^∞ functions with compact support in the interior of Ω_0. This will serve as our set of test functions.

Let $\mathcal{E}$ be a set of elliptic operators E_i of the form

$$E_i = \sum_{j=0}^{k} \sum_{k_1,\cdots,k_j=1}^{n} \overset{i}{a}_{k_1,\cdots,k_j}(x) \frac{\partial^j}{\partial x_{k_1} \cdots \partial x_{k_j}}.$$

Denote the formal adjoint of the operator E_i by E_i^*. Here, and throughout the remainder of this paper, let $\langle\ ,\ \rangle$ denote the $\mathcal{L}^2(\Omega_0)$ inner-product.

We consider the following problem.

Let $\mathcal{U}_0$ be the set of all functions in $\mathcal{U}$ for which:

(i) For some $E\in\mathcal{E}$, $u_t=Eu$ for almost all $(t, x)\in T\times\Omega_0$, where we interpret $u_t=Eu$ to mean $\langle u_t, v\rangle=\langle u, E^*v\rangle$ for all $v\in\mathcal{V}$ and for almost all $t\in T$.

(2.3) (ii) $u(t^0, x)=\phi(x)$ a.e. on Ω_0 where $\phi(x)$ is a given function in $\mathcal{L}^2(\Omega_0)$.

(iii) u and all of its partial derivatives with respect to x of order less than or equal to $k-1$ vanish on the boundary $\partial\Omega_0$ of Ω_0.

Observe that because of (i), the test functions need only depend upon x.

For each $E\in\mathcal{E}$, (2.3) is a standard formulation of a parabolic problem.

We seek a function $u(t, x)$, among all u in $\mathcal{U}_0$, which will minimize a given lower semicontinuous functional $L[u(t^1, x)]$.

Observe that $\mathcal{U}_0$ may be an empty set. For this reason, the nonemptiness of $\mathcal{W}$ is hypothesized in Corollary 5.2.

3. **Preliminaries.** We assume that for all u in $\mathcal{U}$ and all E in $\mathcal{E}$, there is a constant K such that

(3.1) (a) $|\langle Eu, v\rangle| \leqq K\|u\|_k\|v\|_k$ a.e. in $t\in T$ for all $v\in\mathcal{V}$.

(b) For each $v\in\mathcal{V}$ there is a constant K_v, depending upon v, such that $|\int_T E^*v| \leqq K_v$ a.e. in $x\in\Omega_0$.

Note that

$$\langle Eu, v\rangle = \int_{\Omega_0} (u, E^*v) \qquad \text{for } t \text{ in } T.$$

A sequence $\{E_n\}$ in $\mathcal{E}$ is said to be a *weak Cauchy sequence for* fixed

$u \in \mathfrak{U}$ if $\langle (E_n - E_m)u, v\rangle \to 0$ as $m, n \to \infty$ for all $v \in \mathfrak{V}$ almost everywhere in $t \in T$.

A subset $\{E_i\}$ of $\mathcal{E}$ is said to be *weakly sequentially compact* if for each u in $\mathfrak{U}$ there is a sequence $\{E_n\}$ of $\{E_i\}$ which is weakly Cauchy.

The sequence $\{E_n\}$ is said to be *weakly convergent* if there is an $E \in \mathcal{E}$ such that $\langle (E_n - E)u, v\rangle \to 0$ as $n \to \infty$ for all $v \in \mathfrak{V}$ almost everywhere in $t \in T$ and for every $u \in \mathfrak{U}$.

The set $\mathcal{E}$ is said to be *weakly closed* if it is closed with respect to weak convergence.

We shall show in Lemma 4.1 that condition (3.1) (b) assures that $\mathcal{E}$ is weakly sequentially compact but not necessarily closed.

4. **Auxiliary lemmas.** In this section we develop several lemmas which will aid in obtaining the main theorem.

LEMMA 4.1. *Every sequence $\{E_i\}$ in $\mathcal{E}$ contains a subsequence which is weakly Cauchy for each u in $\mathfrak{U}$.*

Let $\mathfrak{U}^*$ and $\mathfrak{V}^*$ be countable subsets of $\mathfrak{U}$ and $\mathfrak{V}$ respectively which are dense in the k-Dirichlet norm. For each $v_j \in \mathfrak{V}^*$, the set of functions

$$\overset{*}{E_i} v_j \qquad \text{for } E_i \in \mathcal{E} \tag{4.1}$$

is weakly sequentially compact as functions of t on T for almost all x in Ω_0 by (3.1) (b). (See [1].)

Hence, for any measurable set $T' \subset T$ and any $u_k \in \mathfrak{U}^*$, the set of functions

$$(\overset{*}{E_i} v_j)u_k \qquad \text{for } E_i \in \mathcal{E} \tag{4.2}$$

is weakly sequentially compact on T' for almost all x in Ω_0. By the usual diagonalization process on j and k, we can choose a subsequence $\{E_n\}$ for which

$$\int_{T'} (\overset{*}{E_n} v)u \tag{4.3}$$

is Cauchy for all v in $\mathfrak{V}^*$, all $u \in \mathfrak{U}^*$, and almost all x in Ω_0. Hence, for this subsequence,

$$\int_{\Omega_0}\int_{T'} [(E_n - E_m)^* v]u \to 0 \quad \text{as } m, n \to \infty. \tag{4.4}$$

Applying Fubini's theorem to (4.4),

$$\int_{T'}\int_{\Omega_0} [(E_n - E_m)^* v]u = \int_{T'} \langle (E_n - E_m)u, v\rangle$$

which approaches 0 as $m, n \to \infty$. Since T' is arbitrary,

$$\langle (E_n - E_m)u, v\rangle \to 0 \qquad \text{as } m, n \to \infty$$

for almost all t in T and u in $\mathcal{U}^*$, v in $\mathcal{V}^*$.

But by (3.1)(a), this must also be true for u in $\mathcal{U}$ and v in $\mathcal{V}$. This completes the proof.

As a consequence of Lemma 4.1 we do not need to specify the choice of u when we refer to a weakly Cauchy sequence $\{E_n\}$.

LEMMA 4.2. *Let $\{E_n\}$ be a weakly Cauchy sequence of $\mathcal{E}$ and suppose that $u_n \to u_0$ in $\mathcal{U}$ in the k-Dirichlet norm. Then a subsequence of $\{E_n u_n\}$ is weakly Cauchy in $\mathcal{L}^2(\Omega_0)$.*

In order to prove this, consider the inequality

$$\begin{aligned} |\langle E_n u_n - E_m u_m, v\rangle| \leqq |\langle E_n(u_n - u_0), v\rangle| + |\langle (E_n - E_m)u_0, v\rangle| \\ + |\langle E_m(u_0 - u_m), v\rangle| . \end{aligned}$$

Using (3.1(a), the right side of the above inequality is less than or equal to

$$K[\|u_n - u_0\|_k + \|u_m - u_0\|_k]\|v\|_k + |\langle (E_n - E_m)u_0, v\rangle| .$$

By Lemma 4.1, a subsequence of the latter term is weakly Cauchy, and by hypothesis, the first term approaches zero as $m, n \to \infty$. The proof is therefore complete.

We note in passing that if E_n converges weakly to E, then $E_n u_n$ converges weakly to Eu_0.

5. **Main theorem.** We proceed in this section to prove the main theorem and to note some of its consequences.

THEOREM 5.1. *If $\mathcal{E}$ is weakly closed, then $\mathcal{U}_0$ is closed in the k-Dirichlet norm.*

Suppose that $\|u_n - u_0\|_k \to 0$ for almost all $t \in T$. Since $\mathcal{U}$ is closed, u_0 is in $\mathcal{U}$. We have, by absolute continuity in t, that

$$\|u_n - u_0\| \to 0 \tag{5.1}$$

for all $t \in T$. Consequently, a subsequence of $u_n(t, x)$, which we denote again by $u_n(t, x)$, converges to $u_0(t, x)$ almost everywhere on Ω_0 for all $t \in T$. In particular,

$$u_n(t^0, x) \to u_0(t^0, x) \qquad \text{a.e. on } \Omega_0 . \tag{5.2}$$

Since each of the partial derivatives with respect to x also converge in the $\mathcal{L}^2(\Omega_0)$ norm, $u_0(t, x)$ satisfies (2.3)(ii) and (2.3)(iii).

Because for every n each u_n is a solution of (2.3)(i), there is an E_n

in $\mathcal{E}$ for which

$$u_{n_t} = E_n u_n. \tag{5.3}$$

Since $\mathcal{E}$ is weakly closed, there is an E in $\mathcal{E}$ for which a subsequence $\{E_n u_n\}$ converges to Eu_0, i.e.,

$$\langle E_n u_n - Eu_0, v\rangle \to 0 \qquad \text{for all } v \text{ in } \mathcal{V}. \tag{5.4}$$

Now, using absolute continuity in t, the initial condition (5.2) and Fubini's theorem, we obtain

$$\left| \int_{[t^0,t]} \langle u_{n_t} - u_{0_t}, v\rangle \right| = \left| \langle u_n - u_0, v\rangle \right| \leqq \|u_n - u_0\| \|v\|.$$

Hence, by (5.1), for almost all t,

$$\langle u_{n_t} - u_{0_t}, v\rangle \to 0 \tag{5.5}$$

for all v in $\mathcal{V}$. Therefore,

$$\begin{aligned} \left| \langle Eu_0 - u_{0_t}, v\rangle \right| \leqq \left| \langle E_n u_n - Eu_0, v\rangle \right| &+ \left| \langle E_n u_n - u_{n_t}, v\rangle \right| \\ &+ \left| \langle u_{n_t} - u_{0_t}, v\rangle \right|. \end{aligned}$$

If we apply (5.3), (5.4) and (5.5) to the above inequality, we obtain

$$u_{0_t} = Eu_0.$$

From this last equation we may conclude that u_0 belongs to $\mathcal{U}_0$, and the theorem follows.

Let $\mathcal{W} = \{u(t^1, x) : u(t, x) \in \mathcal{U}_0\}$. We immediately obtain the following results as corollaries of Theorem 5.1.

COROLLARY 5.1. *If $\mathcal{U}_0$ is closed in the k-Dirichlet norm, then $\mathcal{W}$ is also, and if $\mathcal{U}$ is bounded in the k-Dirichlet norm, then $\mathcal{W}$ is also.*

COROLLARY 5.2. *If $\mathcal{E}$ is weakly closed, if $\mathcal{U}$ is bounded and $\mathcal{W}$ is nonempty, then there is a $u_0 \in \mathcal{U}_0$ for which $L[u(t^1, x)]$ achieves its minimum on $\mathcal{W}$.*

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