

EXTENDING REAL MAPS DEFINED ON A SUBSET OF A DISK

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Let K and Q denote, respectively, the unit circular disk and the unit square disk in E^2 . Let $\Delta: K \rightarrow [0, 1]$ and $\pi: Q \rightarrow [0, 1]$ be defined by $\Delta(x, y) = (x^2 + y^2)^{1/2}$ and $\pi(x, y) = x$. Suppose T is a subset of a disk D and f is a mapping of T into $[0, 1]$; we are concerned here with conditions under which f can be extended to a mapping of D onto $[0, 1]$ which is topologically equivalent to Δ or to π . Recall that a mapping $u: A \rightarrow B$ is *topologically equivalent* to a mapping $w: A' \rightarrow B'$ if there exist homeomorphisms $h: A \rightarrow A'$ and $k: B \rightarrow B'$ such that $ku = wh$.

It was shown in [2] that if T is a compact 0-dimensional subset of $\text{Int } D$, then any mapping f of T into the open interval $(0, 1)$ has an extension to a mapping of D onto $[0, 1]$ which is topologically equivalent to π (and hence also an extension equivalent to Δ). However, the existence of a single arc in T , even though this arc is a subset of a point-inverse of f , may make the desired extension impossible. For example, if T is a subset of $\text{Int } D$ consisting of a horizontal interval A together with points x_1, x_2, x_3, \dots such that the sequence $\{x_{2n}\}$ converges to a midpoint of A from above and $\{x_{2n+1}\}$ converges to a midpoint of A from below, then the mapping f defined by $f(A) = 1/2$, $f(x_n) = 1/2 + 1/3n$ cannot be extended to a mapping equivalent to π . We therefore restrict attention in this paper to the case in which the common part of T and the interior of D is totally disconnected.

First we show that if T is a compact subset of a disk D whose intersection with the interior of D is totally disconnected and T does not contain the boundary of D , then any mapping f of T into $[1/2, 1]$ such that $T \cap \text{Bd } D = f^{-1}(1)$ can be extended to a mapping of D onto $[0, 1]$ which is topologically equivalent to π . Using this result, we show that if T is a compact subset of a disk D whose intersection with the interior of D is totally disconnected and T contains the boundary of D , then any mapping f of T into $[1/2, 1]$ such that $\text{Bd } D = f^{-1}(1)$ can be extended to a mapping of D onto $[0, 1]$ which is topologically equivalent to Δ .

It follows as a special case of results in [1] that these theorems do not directly generalize to higher dimensions. If $n \geq 3$, $1 \leq k \leq n$, and

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π' denotes the projection onto the last k coordinates of points of I^n , there is a Cantor set A in $\text{Int } I^n$ such that $\text{Int } I^n - A$ is not simply connected. If f is the constant map from A to $p = (1/2, 1/2, \dots, 1/2)$ in $\text{Int } I^k$, then there is no extension of f to all of I^n that is equivalent to π' . To see this, suppose that f' is a mapping from I^n onto I^k extending f for which there are self homeomorphisms h and g of I^n and I^k respectively such that $gf' = \pi'h$. Then $h(A)$ would lie in $\pi'^{-1}(g(p))$, and if q denotes the projection given by $q(x_1, x_2, \dots, x_n) = (x_2, x_3, \dots, x_n)$ in I^{n-1} , then $qh(A)$ is a Cantor set in $\text{Int } I^{n-1}$, and by Theorem 3E of [1], $\text{Int } I^n - h(A)$ is simply connected—a contradiction.

The following lemmas are used frequently in the construction which follow and so are stated separately.

LEMMA 1. *If S is a compact 0-dimensional subset of E^2 covered by a collection \mathcal{G} of open sets, then there is a refinement of \mathcal{G} of mutually exclusive disks whose interiors cover S .*

This follows readily from the fact that every Cantor set in E^2 is tame.

DEFINITION. The arc (or disk) A spans the connected domain G from x to y means that x and y are two boundary points of G , $A - \{x, y\}$ is a connected subset of G , and $A \cap \text{Bd } G = \{x, y\}$.

LEMMA 2. *If T is a compact subset of Q such that $T \cap \text{Int } Q$ is 0-dimensional and D is a disk in Q with boundary points x and y , then there exists a disk which spans $\text{Int } D$ from x to y which does not intersect $T - \{x, y\}$.*

INDICATION OF PROOF. It is not difficult to construct an arc A spanning $\text{Int } D$ from x to y and missing $T - \{x, y\}$. If A is such an arc, there is a disk D' containing A such that $D' - \{x, y\}$ is a subset of $(\text{Int } D) - T$, and hence D' spans $\text{Int } D$ from x to y .

THEOREM 1. *If T is a compact subset of a disk D , $\text{Bd } D \not\subset T$, and $T \cap \text{Int } D$ is totally disconnected, then every mapping of T into $[1/2, 1]$ whose inverse of 1 is $T \cap \text{Bd } D$ can be extended to a mapping of D onto $[0, 1]$ which is equivalent to π .*

PROOF. If T and D are as in the hypothesis, then there is a self homeomorphism v of E^2 such that $v(D)$ is Q and $V(T \cap \text{Bd } D)$ is contained in the right vertical edge $A = \{(1, y) \mid 0 \leq y \leq 1\}$. Let T' denote $A \cup v(T)$, and suppose f is a mapping of T into $[1/2, 1]$ such that $f^{-1}(1) = T \cap \text{Bd } D$. It will be shown that there exists a self homeo-

morphism w of Q such that for each t in T' , $\pi w(t) = f(v^{-1}(t))$. From this it will follow that $\pi w v$ is an extension of f that is equivalent to π . Denote by f' the composition $f v^{-1}$.

There is a disk D_1 spanning $\text{Int } Q$ from $(0, 1/2)$ to $(1, 1/2)$ which contains only the point $(1, 1/2)$ of T' ; and there is an arc L_1 spanning $\text{Int } Q$ from $(3/4, 0)$ to $(3/4, 1)$ which misses T' , intersects the boundary of D_1 in exactly two points, and lies entirely to the right of the vertical line through $(1/2, 0)$. Clearly L_1 divides Q into two disks; denote the one containing A by L'_1 , and if E_1 denotes $(Q - \text{Int } D_1) \cap L'_1$, then E_1 contains $T' \cap L'_1$, $\text{Bd } E_1 \cap T' = A$, the closure of each component of $\text{Int } E_1$ is a disk of diameter less than 1, and A is accessible from the domain $\text{Int } Q - E_1$.

Recall that "the point set A is accessible from the domain G " means that for some point x in A , there is an arc in $G \cup \{x\}$ which contains x .

There are disks D_2 and D_3 such that (1) D_2 spans the lower component of $\text{Int } E_1$ from a point of L_1 to $(1, 1/4)$ and D_3 spans the upper component from a point of L_1 to $(1, 3/4)$, and (2) D_2 contains only the point $(1, 1/4)$ of T' and D_3 the point $(1, 3/4)$ of T' . The boundary of $D_1 \cup D_2 \cup D_3$ consists of two arcs emanating from $(1, 1/4)$ plus two arcs from $(1, 1/2)$ plus two from $(1, 3/4)$. Hence there is a vertical line V between L_1 and A such that each component of $\text{Int } Q - (V \cup \text{Bd}(D_1 \cup D_2 \cup D_3))$ to the right of V has diameter less than $1/2$. There is an arc L_2 spanning $\text{Int } Q$ from top to bottom between V and A that misses T' and intersects each of $\text{Bd } D_1$, $\text{Bd } D_2$ and $\text{Bd } D_3$ in exactly two points, and such that L_2 lies to the right of the vertical line through $(3/4, 0)$. Again, L_2 divides Q into two cells, L'_2 denotes the one containing A , and if E_2 denotes $[E_1 - \text{Int}(D_2 \cup D_3)] \cap L'_2$, then $E_2 \subset E_1$, E_2 contains $T' \cap L'_2$, $\text{Bd } E_2 \cap T' = A$, A is accessible from each of the two components of $(\text{Int } E_1) - E_2$, and the closure of each component of $\text{Int } E_2$ is a disk of diameter less than $1/2$.

In this way, a sequence $\{E_i\}$ is constructed such that for each i :

(1) E_i is the union of 2^i disks each of which has diameter less than $1/2^{i-1}$ and intersects A in an arc, such that if two of these disks intersect, then the intersection is a single point of A ,

(2) $(\text{Bd } E_i) \cap T' = A$,

(3) $E_i \subset E_{i-1}$,

(4) there is an arc spanning $\text{Int } Q$ between the vertical line through $(1 - 1/2^{i+1}, 0)$ and A such that E_i contains in its interior every point of T' to the right of this arc, and

(5) if $i > j \geq 0$, then A is accessible from each of the 2^j components of $(\text{Int } E_j) - E_i$.

For each i , denote by H_i the collection of closures of components of $\text{Int } E_i$. Let G_i denote a finite collection of mutually exclusive disks in $(\text{Int } Q) - E_i$ whose interiors cover $T' - E_i$. If G_i has been defined, define G_{i+1} to be a collection of mutually exclusive disks in $(\text{Int } Q) - E_{i+1}$ whose interiors cover $T' - E_{i+1}$ such that each disk has diameter less than $1/2^{i+1}$ and lies in the interior of some element of $G_i \cup H_i$.

Since f' is a continuous mapping on the compact set T and mesh $G_i \cup H_i \rightarrow 0$ as $i \rightarrow \infty$, for each $\epsilon > 0$ there is an integer n such that $f'(k \cap T') < \epsilon$ for each $k \in G_n \cup H_n$. Hence it may be assumed that for each n , the diameter of $f'(k \cap T')$ is less than $1/2^{n+1}$ if k is in $G_n \cup H_n$.

There is a self homeomorphism u_1 of Q such that

- (1) if $c \in G_1$, then $u_1(c)$ is a rectangular disk of diameter less than $1/2$ with sides parallel to the coordinate axes,
- (2) if $b \in H_1$, then $u_1(b)$ is a quadrilateral disk of diameter less than $1/2$,
- (3) if $c \in G_1 \cup H_1$, then $\pi(\text{Int } u_1(c))$ contains $f'(c \cap T')$, and
- (4) u_1 is the identity on $\text{Bd } Q$.

Suppose for each $i < m$, u_i has been defined. Let w_m denote the composition $u_{m-1} \cdot u_{m-2} \cdots u_2 \cdot u_1$. There is a self homeomorphism u_m of Q such that

- (1) if $c \in G_m$, then $u_m w_m(c)$ is a rectangular disk of diameter less than $1/2^m$ with sides parallel to the coordinate axes,
- (2) if $b \in H_m$, then $u_m w_m(b)$ is the interior of a quadrilateral disk of diameter less than $1/2^m$,
- (3) if $c \in G_m \cup H_m$, then $\pi(\text{Int } u_m w_m(c))$ contains $f'(c \cap T')$, and
- (4) u_m is the identity on $\text{Bd } Q \cup (Q - \cup \{w_m(c) \mid c \in G_{m-1} \cup H_{m-1}\})$.

This defines the sequence $\{w_i\}$ of self homeomorphisms of Q . Since u_i moves no point more than $1/2^{i-1}$, $\{w_i\}$ is Cauchy and converges to a continuous function w of Q onto Q . By condition (3) on each u_i , $\pi w(x) = f'(x)$ for all x in T' . If there are an x and y in Q such that $w(x) = w(y)$, then for each i , $w_i(x)$ and $w_i(y)$ must be (if unequal) in the same quadrilateral disk or rectangular disk, as described in properties (2) and (1) of u_i . Hence $w_{i+1}^{-1}(x)$ and $w_{i+1}^{-1}(y)$ are both in the same element of $G_i \cup H_i$. Hence, $d(x, y) \leq 1/2^i$ so that $x = y$. Hence w is a homeomorphism and since $\pi w(t) = f'(t) = f v^{-1}(t)$ for $t \in T'$, w has the required properties. Because it is needed in the proof of Theorem 2, note that $w(x) = x$ for all $x \in \text{Bd } Q$.

THEOREM 2. *If T is a compact subset of a disk D such that $\text{Bd } D \subset T$ and $T \cap \text{Int } D$ is totally disconnected, then every map from T into $[1/2, 1]$ whose inverse of 1 is $\text{Bd } D$ can be extended to a mapping of D onto $[0, 1]$ which is equivalent to Δ .*

PROOF. As in the proof of Theorem 1, there is no loss in generality if D is assumed to be the unit circular disk centered at the origin. Also, assume that the circular disk D' centered at the origin with radius $1/3$ contains no point of T . Let B_1 and B_2 be disjoint arcs irreducible from $\text{Bd } D'$ to $\text{Bd } D$ that do not intersect $T \cap \text{Int } D$. There is a homeomorphism of D onto itself that takes $B_1 \cup B_2$ onto two straight line segments each a subset of a radial line in D , so there is no loss in generality in assuming that B_1 and B_2 are themselves each a subset of a radial line in D . If D_1 and D_2 are the disks into which $D - \text{Int } D'$ is divided by $B_1 \cup B_2$, it follows from Theorem 1 that there exist homeomorphisms u_1 of D_1 onto itself and u_2 of D_2 onto itself such that for $i = 1, 2$,

(1) u_i is the identity on $\text{Bd } D_i$ and

(2) for each x in $[1/3, 1)$, $u_i(f^{-1}(x) \cap D_i)$ is a subset of the circle with center $(0, 0)$ and radius x . There is a homeomorphism u of D onto itself which is an extension of u_1 and u_2 and is the identity on D' . The mapping Δu is an extension of f which is equivalent to Δ .

Theorem 2 may be applied to give results on the problem of embedding collections of compact, totally disconnected subsets of the plane in collections of arcs or simple closed curves. One such theorem is the following.

COROLLARY. *Suppose M_1, M_2, M_3, \dots is a sequence of compact, totally disconnected subsets of E^2 converging to a nondegenerate compact set M which does not intersect $\cup M_i$. Then there is a homeomorphism of E^2 onto itself which takes each M_i into a different circle with center $(0, 0)$ if and only if M is a subset of a simple closed curve which does not intersect $\cup M_i$ and does not separate two points of any M_i , and there is a homeomorphism of E^2 onto itself which takes each M_i into a different horizontal interval if and only if M is a proper subset of such a simple closed curve.*

PROOF. The necessity of each condition is clear.

Suppose M is a subset of a simple closed curve C and that each M_i is a subset either of $\text{Int } C$ or of $\text{Ext } C$. Assume that C is the unit circle, and let I and \bar{E} denote, respectively, the interior and the exterior of C .

It follows from Theorem 2 that there is a homeomorphism h_1 of the plane onto itself which is the identity on E and which takes each M_i lying in I into a circle with center $(0, 0)$. Applying Theorem 2 to the one-point compactification of E , it can be shown that there is a homeomorphism h_2 of the plane onto itself which is the identity on I and which takes each M_i lying in E into a circle with center $(0, 0)$. The

homeomorphism $h = h_1 h_2$ takes each M_i into a circle with center $(0, 0)$; if M is a proper subset of C , h can be modified so as to take each M_i into a horizontal interval.

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