POINTED MINIMAL SETS AND S-REGULARITY

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Let (X, T) be a point transitive transformation group $(\mathrm{Cl}(xT)=X$ for some $x\in X)$ with compact Hausdorff phase space X and discrete phase group T. Let βT be the Stone-Čech compactification of T. If $t\in T$ and $\{s_m\}$ is a net in T converging to $p\in\beta T$, put $pt=\lim s_m t$. If further $\{t_n\}$ is a net in T converging to $q\in\beta T$, put $pq=\lim pt_n$. Ellis has observed [3] that these definitions make $(\beta T, T)$ into a transformation group and make βT into a semigroup in which left multiplication is continuous but right multiplication is generally not.

Let $\mathcal{C}(\beta T)=\mathcal{C}$ be the algebra of all real-valued continuous functions on βT. For $f\in\mathcal{C}$ and $t\in T$ let fR_t be the element of $\mathcal{C}$ given by $\langle fR_t, p\rangle =\langle f, pt\rangle$ $(p\in\beta T)$ (generally $\langle f, p\rangle=pf=$ the image of p under f). Define a subalgebra $\mathfrak{A}$ of $\mathcal{C}$ to be a *T-subalgebra* if it is uniformly closed and if $fR_t\in\mathfrak{A}$ whenever $f\in\mathfrak{A}$ and $t\in T$. Call a homomorphism of $\mathfrak{A}$ into $\mathcal{C}$ a *T-homomorphism* if it commutes with R_t for all $t\in T$, and denote the set of all T-homomorphisms on $\mathfrak{A}$ by $|\mathfrak{A}|$.

For each $f\in\mathfrak{A}$ and $t\in T$, define the map σ_{ft} of $|\mathfrak{A}|$ into the reals by $\phi\sigma_{ft}=\langle f\phi, t\rangle$ $(\phi\in|\mathfrak{A}|)$. Ellis has shown [4] that, if $|\mathfrak{A}|$ is provided with the smallest topology such that all the maps σ_{ft} are continuous, then the action defined by $\langle f(\phi t), p\rangle=\langle f\phi, tp\rangle$ $(f\in\mathfrak{A}, \phi\in|\mathfrak{A}|, t\in T, p\in\beta T)$ makes $(|\mathfrak{A}|, T)$ into a point transitive transformation group for each T-subalgebra $\mathfrak{A}$. Further, given (X, T), a T-subalgebra $\mathfrak{A}$ can be found such that $(|\mathfrak{A}|, T)$ is isomorphic to (X, T) (we say $\mathfrak{A}$ *corresponds* to X). In general there are many T-subalgebras corresponding to a given X. For convenience we repeat the Ellis construction, without proof. Choose $x\in X$ such that $\mathrm{Cl}(xT)=X$. The map π_x of T onto X given by $t\pi_x=xt$ $(t\in T)$ becomes, when extended continuously, a homomorphism π_x^* of $(\beta T, T)$ onto (X, T); that is, a continuous map commuting with T. Write xp for $p\pi_x^*$ $(p\in\beta T)$, and define a map x^* from $\mathcal{C}(X)$ into $\mathcal{C}$ by $\langle fx^*, p\rangle=\langle f, xp\rangle$ $(f\in\mathcal{C}(X), p\in\beta T)$. Then $\mathfrak{A}_x =\mathcal{C}(X)x^*$ corresponds to X. Further, there is a natural isomorphism between $(|\mathfrak{A}_x|, T)$ and (X, T) taking the inclusion map of $|\mathfrak{A}_x|$ onto x.

Following Ellis, we shall say that a T-subalgebra $\mathfrak{A}$ of $\mathcal{C}$ has a certain recursive property if $(|\mathfrak{A}|, T)$ has that property. We shall primarily consider minimal algebras. For $f\in\mathcal{C}$, $q\in\beta T$, define $fq\in\mathcal{C}$ by $\langle fq, p\rangle=\langle f, qp\rangle$ $(p\in\beta T)$. Then Ellis has shown [5] that $\mathfrak{A}$ is mini-

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mal if and only if $\mathcal{A} \subset \{f \in \mathcal{C} \mid fu = f\}$ for some idempotent u in a (universal) minimal subset M of βT.

The study of minimal sets by means of their corresponding algebras is frequently complicated by the one to many nature of the correspondence. Our first objective will be to investigate conditions on x, $y \in X$ so that $\mathcal{A}_x = \mathcal{A}_y$.

For the remainder of this paper M will denote an arbitrary but fixed minimal set in βT and u will denote an arbitrary but fixed idempotent in M.

THEOREM 1. *Let $x \in X$. Then $xu = x$ if and only if $fu = f$ for all $f \in \mathcal{A}_x$.*

PROOF. First, assume $xu = x$. Select $f \in \mathcal{A}_x$. Then $f = gx^*$ for some $g \in \mathcal{C}(X)$. Thus for all $t \in T$ we have $\langle f, t\rangle = \langle gx^*, t\rangle = \langle g, xt\rangle = \langle g, (xu)t\rangle = \langle g, x(ut)\rangle = \langle gx^*, ut\rangle = \langle (gx^*)u, t\rangle = \langle fu, t\rangle$. Thus $f = fu$ on T. But f and fu are continuous and T is dense in βT. Thus $f = fu$ on βT.

Conversely, assume $fu = f$ for all $f \in \mathcal{A}_x$. Then for all $g \in \mathcal{C}(X)$ we have $\langle g, xu\rangle = \langle gx^*, u\rangle = \langle (gx^*)u, e\rangle = \langle gx^*, e\rangle = \langle g, x\rangle$ where e is the identity element of T. But then $x = xu$ since X is compact Hausdorff whence $\mathcal{C}(X)$ separates points.

DEFINITION 1. Let N be any minimal set in βT, and let v be any idempotent in N. Denote by K_v the category given by

$$\mathrm{Ob}(K_v) = \{\mathcal{A} \mid f = fv\ (f \in \mathcal{A})\} \text{ and}$$
$$\mathrm{Hom}\,(\mathcal{A}, \mathcal{B}) = \{p \in \mid \mathcal{A} \mid \ \mid \mathcal{A}p \subset \mathcal{B}\}\ (\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(K_v)).$$

A proof of the following theorem appears in [5].

THEOREM 2. *Let N and L be two not necessarily distinct minimal sets in βT, and let v and η be idempotents in N and L respectively. Then the categories K_v and K_η are equivalent. If λ is the idempotent in L such that $v\lambda = v$ and $\lambda v = \lambda$ (see [2]), the equivalence F is given by $\mathcal{A}F = \mathcal{A}\eta$ ($\mathcal{A} \in \mathrm{Ob}(K_v)$) and $pF = \lambda p \eta$ ($p \in \mathrm{Hom}(K_v)$).*

For the remainder of this paper, in addition to the standing assumptions noted earlier, we let (X, T) be a minimal set.

DEFINITION 2. For each $x \in X$ we shall call the ordered triple (X, x, T) a *pointed minimal set.* Ordinarily we shall write (X, x) for (X, x, T).

We shall say the algebra $\mathcal{A}_x$ constructed earlier *corresponds* to (X, x).

DEFINITION 3. A *homomorphism* (*isomorphism*) from (X, x) to (Y, y) is a homomorphism (isomorphism) π from (X, T) to (Y, T) such that $x\pi = y$.

Notice that (X, x) and $(|\mathfrak{A}_x|, i)$ are isomorphic, where i is the inclusion map of $\mathfrak{A}_x$ into $\mathfrak{C}$.

THEOREM 3. *If a homomorphism exists from* (X, x) *to* (Y, y) *it is unique.*

PROOF. Let π be such a homomorphism. Let $t\in T$. Then $(xt)\pi = (x\pi)t = yt$, so π is determined on xT. By continuity π is determined on $\mathrm{Cl}(xT) = X$.

DEFINITION 4. Let T_u be the category given by

$$\mathrm{Ob}(T_u) = \{(|\mathfrak{A}|, T) \mid f = fu \ (f \in \mathfrak{A})\}$$

and $\mathrm{Hom}((|\mathfrak{A}|, T), (|\beta|, T))$ = the set of homomorphisms from $(|\mathfrak{A}|, T)$ to $(|\mathfrak{B}|, T)$ $(\mathfrak{A}, \mathfrak{B}\in\mathrm{Ob}(T_u))$.

THEOREM 4. *The categories* T_u *and* K_u *are equivalent.*

PROOF. For $\mathfrak{A}\in\mathrm{Ob}(K_u)$ define $\mathfrak{A}F = (|\mathfrak{A}|, T)$. For $\mathfrak{A}, \mathfrak{B}\in\mathrm{Ob}(K_u)$ and $p\in\mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$, define $pF\in\mathrm{Hom}((|\mathfrak{B}|, T), (|\mathfrak{A}|, T))$ by $\langle pF, q\rangle = pq$ $(q\in|\mathfrak{B}|)$. Then F is a contravariant functor from K_u to T_u. To show that the law of composition works as it should, select $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{F}\in\mathrm{Ob}(K_u)$, $p\in\mathrm{Hom}(\mathfrak{A}, \mathfrak{B})$, and $r\in\mathrm{Hom}(\mathfrak{B}, \mathfrak{F})$. Then for any $q\in|\mathfrak{F}|$, $\langle rF, q\rangle = rq$, and $\langle pF, rq\rangle = prq = \langle (pr)F, q\rangle$. That is, $(rF)(pF) = (pr)F$.

Trivially F is a one-one map of $\mathrm{Ob}(K_u)$ onto $\mathrm{Ob}(T_u)$. By [**4**, Proposition 2] F is a one-one map of $\mathrm{Hom}(K_u)$ onto $\mathrm{Hom}(T_u)$. Thus F is an equivalence.

DEFINITION 5. Let S_u be the category given by $\mathrm{Ob}(S_u) = \{(X, x) \mid xu = x\}$ and $\mathrm{Hom}((X, x), (Y, y))$ = the set of homomorphisms from (X, T) to (Y, T) $((X, x), (Y, y)\in\mathrm{Ob}(S_u))$.

THEOREM 5. *There exists a contravariant functor* H *taking* S_u *onto* K_u *such that* $(X, x)H = \mathfrak{A}_x$ *for all* $(X, x)\in\mathrm{Ob}(S_u)$.

PROOF. This follows directly from Theorem 4 and the remark after Definition 3 if we identify the pointed minimal set $(|\mathfrak{A}_x|, i)$ with the object $(|\mathfrak{A}_x|, T)$ of T_u.

LEMMA 1. *With the notation of Theorem* 2, *let* $\mathfrak{A}, \mathfrak{B}\in\mathrm{Ob}(K_v)$ *be such that* $\mathfrak{A}\subset B$, *and let* i *be the inclusion map of* $\mathfrak{A}$ *into* $\mathfrak{B}$. *Then* iF *is the inclusion map of* $\mathfrak{A}\eta$ *into* $\mathfrak{B}\eta$.

PROOF. Select $g\in\mathfrak{A}\eta$. Then $g = f\eta$ for some $f\in\mathfrak{A}$. With λ as in Theorem 2, $g\lambda i\eta = f\eta\lambda i\eta = fv\eta\lambda i\eta = fv\lambda i\eta = fvi\eta = fi\eta = f\eta = g$.

LEMMA 2. *Let* F *be the functor from* K_u *to* T_u *defined in the proof of*

Theorem 4. *Let $\mathfrak{A}$, $\mathfrak{B}\in \mathrm{Ob}(K_u)$ be such that $\mathfrak{A}\subset\mathfrak{B}$, and let i be the inclusion map of $\mathfrak{A}$ into $\mathfrak{B}$. Then iF is the restriction map from $|\mathfrak{B}|$ to $|\mathfrak{A}|$.*

PROOF. Follows immediately from the definition of F.

DEFINITION 6. Let v be any idempotent in any minimal set of βT. Let K_v' be the category given by $\mathrm{Ob}(K_v')=\mathrm{Ob}(K_v)$, $\mathrm{Hom}(\mathfrak{A},\mathfrak{B})=$ the inclusion map whenever $\mathfrak{A}$, $\mathfrak{B}\in\mathrm{Ob}(K_v')$ and $\mathfrak{A}\subset\mathfrak{B}$, and $\mathrm{Hom}(\mathfrak{A},\mathfrak{B})=\emptyset$ otherwise.

DEFINITION 7. Let S_u' be the category given by $\mathrm{Ob}(S_u')=\mathrm{Ob}(S_u)$, $\mathrm{Hom}((X,x),(Y,y))=$ the homomorphism from (X,x) to (Y,y) if it exists, and $\mathrm{Hom}((X,x),(Y,y))=\emptyset$ otherwise.

THEOREM 6. *S_u' and K_u' are equivalent. If v and η are any idempotents in any minimal sets of βT, then K_v' and K_η' are equivalent.*

PROOF. It is known [4] that whenever $\mathfrak{A}\subset\mathfrak{B}$ the restriction map from $|\mathfrak{B}|$ to $|\mathfrak{A}|$ is a homomorphism of $(|\mathfrak{B}|, T)$ to $(|\mathfrak{A}|, T)$. Thus the result follows immediately from Lemmas 1 and 2 and the definitions involved.

COROLLARY 1. *Let $\mathfrak{B}_y$ be the algebra corresponding to the pointed minimal set (Y,y). Assume $xu=x$ and $yu=y$. Then (Y,y) is a homomorphic image of (X,x) if and only if $\mathfrak{B}_y\subset\mathfrak{A}_x$.*

COROLLARY 2. *Choose x, $y\in X$ such that $xu=x$ and $yu=y$. Then (X,x) and (X,y) correspond to the same algebra if and only if they are isomorphic.*

These results show clearly that when we study the class of all T-subalgebras $\mathfrak{A}$ of $\mathfrak{C}$ such that $f\in\mathfrak{A}$ implies $fu=f$, we are really studying the class of all pointed minimal sets (X,x) such that $xu=x$, with isomorphic sets identified.

In [1] Auslander gives the following definition: a minimal set (X,T) is *regular* if for every almost periodic point (y,z) of $(X\times X,T)$ there exists a homomorphism of (X,y) onto (X,z). He then constructs a lattice of regular minimal sets. The algebraic approach enables us to generalize this lattice and to explain its "niceness." We first review some definitions in [5].

DEFINITION 8 [ELLIS]. Set $G=\{p\in M\,|\,pu=p\}$. If H is any subgroup of G, set $\mathfrak{A}(H)=\{f\in\mathfrak{C}\,|\,f\alpha=f(\alpha\in H)\}$. If $\mathfrak{A}$ is any T-subalgebra of $\mathfrak{A}(u)$, set $G(\mathfrak{A})=\{\alpha\in G\,|\,f\alpha=f\ (f\in\mathfrak{A})\}$.

DEFINITION 9. Let S be any subsemigroup of G, and let $\mathfrak{A}$ be any T-subalgebra of $\mathfrak{A}(u)$. We say $\mathfrak{A}$ is *S-regular* if $\mathfrak{A}\alpha\subset\mathfrak{A}$ for all $\alpha\in S$.

If $\mathfrak{A}$ is any subset of $\mathfrak{C}$ we shall write $\{\mathfrak{A}\}$ for the smallest T-subalgebra containing $\mathfrak{A}$.

LEMMA 3. *Let Ω be an index set. For each $\omega \in \Omega$ let $\mathfrak{A}_\omega$ be a T-subalgebra of $\mathfrak{A}(u)$. If each $\mathfrak{A}_\omega$ is S-regular, then so are $\{\bigcup_\omega \mathfrak{A}_\omega\}$ and $\bigcap_\omega \mathfrak{A}_\omega$.*

PROOF. Let $\alpha \in S$ and let $g \in \bigcup_\omega \mathfrak{A}_\omega$. Pick $\lambda \in \Omega$ so $g \in \mathfrak{A}_\lambda$. Then $g\alpha \in \mathfrak{A}_\lambda \subset \bigcup_\omega \mathfrak{A}_\omega$. Thus $(\bigcup_\omega \mathfrak{A}_\omega)\alpha \subset \bigcup_\omega \mathfrak{A}_\omega$, whence $\{\bigcup_\omega \mathfrak{A}_\omega\}\alpha \subset \{\bigcup_\omega \mathfrak{A}_\omega\}$.

Let $f \in \bigcap_\omega \mathfrak{A}_\omega$. For each $\lambda \in \Omega$, $f \in \mathfrak{A}_\lambda$ and thus $f\alpha \in \mathfrak{A}_\lambda$. Hence $f\alpha \in \bigcap_\omega \mathfrak{A}_\omega$.

The next theorem follows immediately.

THEOREM 7. *Under the operations $\mathfrak{A} \wedge \mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}$ and $\mathfrak{A} \vee \mathfrak{B} = \{\mathfrak{A} \cup \mathfrak{B}\}$, the class of all S-regular T-subalgebras of $\mathfrak{A}(u)$ forms a complete lattice $\mathfrak{L}_S(T)$.*

DEFINITION 10. Let (X, x) be a pointed minimal set with $xu = x$. We say (X, x), is *S-regular* if, given $\alpha \in S$, there exists a homomorphism π from (X, x) to $(X, x\alpha)$.

THEOREM 8. *(X, x) is S-regular if and only if $\mathfrak{A}_x$ is S-regular.*

PROOF. By Corollary 1, π exists if and only if $\mathfrak{A}_{x\alpha} \subset \mathfrak{A}_x$. But $\mathfrak{A}_{x\alpha} = \mathfrak{A}_x\alpha$ by construction. Thus π exists for all $\alpha \in S$ if and only if $\mathfrak{A}_x\alpha \subset \mathfrak{A}_x$ for all $\alpha \in S$, and the theorem follows.

THEOREM 9. *Define a partial ordering on the class $\mathfrak{M}_S(T)$ of all S-regular pointed minimal sets (identified up to isomorphism) by $(Y, y) \geqq (X, x)$ if there exists a homomorphism of (Y, y) onto (X, x). Then $\mathfrak{M}_S(T)$ is a complete lattice, and, in fact, $\mathfrak{M}_S(T)$ is isomorphic to $\mathfrak{L}_S(T)$.*

PROOF. This follows immediately from Theorem 4 and Corollary 1.

The two extreme cases are of particular interest to us. First, if $S = \{u\}$ we observe that the conditions of Definitions 9 and 10 always hold trivially, so that $\mathfrak{M}_u(T)$ and $\mathfrak{L}_u(T)$ are in effect the lattices of all pointed minimal sets and of all minimal algebras respectively. Gottschalk has studied $\mathfrak{M}_u(T)$ from a somewhat different point of view [7], calling its members "ambits."

At the other extreme, putting $S = G$, we recapture Auslander's lattice $\mathfrak{R}(T)$.

LEMMA 4. *Let X be a minimal set, and let $x = xu$ and $y = yu$ in X. Then there exists $\alpha \in G$ such that $x\alpha = y$.*

PROOF. Since $\mathrm{Cl}\,(xT) = X$, we can find $p \in \beta T$ such that $xp = y$. But then $xupu = xpu = yu = y$, and $up \in M$ (see [2]), whence $upu \in G$.

LEMMA 5. *If (X, x) is G-regular for some x such that $xu=x$, then (X, x) is G-regular for all x such that $xu=x$.*

PROOF. If (X, x) is G-regular then $\mathfrak{A}_x\alpha\subset\mathfrak{A}_x$ for all $\alpha\in G$. Let $y=yu$. Choose $\alpha\in G$ so $x\alpha=y$. Then $\mathfrak{A}_y=\mathfrak{A}_x\alpha\subset\mathfrak{A}_x$ and $\mathfrak{A}_x=\mathfrak{A}_x\alpha^{-1}\alpha \subset\mathfrak{A}_x\alpha=\mathfrak{A}_y$, so $\mathfrak{A}_y=\mathfrak{A}_x$ and is G-regular.

THEOREM 10. *(X, x) is G-regular if and only if X is regular.*

PROOF. If X is regular and $\alpha\in G$, then $x\alpha=x\alpha u$, so $(x, x\alpha)$ is an almost periodic point of $X\times X$. By the definition of regular we can find π as required in Definition 10.

Conversely, if (X, x) is G-regular and (y, z) is an almost periodic point of $X\times X$, choose an idempotent w so $y=yw$ and $z=zw$. Let v be the idempotent in M such that $vw=v$ and $wv=w$ (see [2]), and let π be the homomorphism from (X, yu) to (X, zu). Then $y\pi=yw\pi =ywv\pi=yv\pi=yuv\pi=yu\pi v=zuv=zv=zwv=zw=z$.

COROLLARY 3. *The minimal set (X, T) is regular if and only if it has a unique corresponding algebra in $\mathfrak{A}(u)$.*

PROOF. This follows immediately from Theorem 10 and Corollary 2.

From the algebraic point of view the nature and relationship of the lattices considered by Auslander and Gottschalk becomes quite clear. There is a lattice of algebras $\mathfrak{L}_u(T)$, with a family of sublattices $\mathfrak{L}_S(T)$, in which union and intersection replace the considerably more involved constructions in [1]. From the algebraic viewpoint there is nothing special or "nice" about $\mathfrak{L}_G(T)$, other than its being the smallest sublattice in this family. But Corollary 3 tells us exactly that $\mathfrak{L}_G(T)$ is precisely that lattice for which the translation back to the conventional topological point of view is "nice."

We conclude by presenting explicitly the constructions of inf and sup in $\mathfrak{M}_S(T)$.

THEOREM 11. *Let Ω be an index set. For each $\omega\in\Omega$ let (X_ω, T) be minimal and let (X_ω, x_ω) be S-regular. Define $Y=\mathrm{Cl}((x_\omega)T)\subset\prod_\omega X_\omega$. Then $(Y, (x_\omega))=\bigvee_\omega (X_\omega, x_\omega)$.*

PROOF. Let $\alpha\in S$, and let π_ω be the homomorphism of (X_ω, x_ω) onto $(X_\omega, x_\omega\alpha)(\omega\in\Omega)$. If $y=(y_\omega)\in Y$, define π on Y by $y\pi=(y_\omega\pi_\omega)$. Then π is a homomorphism of $(Y, (x_\omega))$ onto $(Y, (x_\omega)\alpha)=(Y, (x_\omega\alpha))$, whence $(Y, (x_\omega))$ is S-regular.

For each $\omega\in\Omega$, projection of Y onto X_ω takes (x_ω) to x_ω. Thus $(Y, (x_\omega))\geqq(X_\omega, x_\omega)$.

Now suppose there exists an S-regular pointed minimal set (W, w) such that, given $\omega \in \Omega$, we have $(W, w)\pi_\omega = (X_\omega, x_\omega)$. Define π on W by $z\pi = (z\pi_\omega)$ $(z \in W)$. Then $w\pi = (x_\omega)$. Since (W, T) is minimal the image of W in $\prod_\omega X_\omega$ must be minimal and therefore must equal $\mathrm{Cl}\,((x_\omega)T) = Y$.

THEOREM 12. *With the notation of Theorem* 11, *define* $\mathfrak{F} \subset \mathcal{C}(Y)$ *by*: $f \in \mathfrak{F}$ *if* f *factors through each* $X_\omega(\omega \in \Omega)$. *Define* $Q \subset Y \times Y$ *by*: $Q = \{(y, z) \mid yf = zf\ (f \in \mathfrak{F})\}$. *Then* $(Y, (x_\omega))/Q = \bigwedge_\omega (X_\omega, x_\omega)$.

PROOF. Let $\alpha \in S$, and let π be the endomorphism on Y such that $(x_\omega)\pi = (x_\omega)\alpha$. To show $(Y, (x_\omega))/Q$ is S-regular, it will suffice to show that $(y, z) \in Q$ implies $(y\pi, z\pi) \in Q$.

In fact, suppose $(y, z) \in Q$ and $f \in \mathfrak{F}$. Let ϕ_ω be the projection of Y onto X_ω $(\omega \in \Omega)$. For each $\omega \in \Omega$ there exists $g_\omega \in \mathcal{C}(X_\omega)$ so $f = \phi_\omega g_\omega$. Then $\pi f = \pi\phi_\omega g_\omega$, so $\pi f \in \mathfrak{F}$. Thus $y\pi f = z\pi f$, or $(y\pi, z\pi) \in Q$.

Next, pick $\lambda \in \Omega$. We show $(X_\lambda, x_\lambda) \geqq (Y, (x_\omega))/Q$.

Choose $y, z \in Y$. If $y\phi_\lambda = z\phi_\lambda$ we claim $(y, z) \in Q$. For, picking $f \in \mathfrak{F}$, there exists $g \in \mathcal{C}(X_\lambda)$ such that $yf = y\phi_\lambda g = z\phi_\lambda g = zf$. Thus we can define the homomorphism h of (X_λ, x_λ) onto $(Y, (x_\omega))/Q$ by $(y\phi_\lambda)h = yQ$.

Finally, suppose there exists an S-regular (W, w) such that, for each $\omega \in \Omega$, we have $(X_\omega, x_\omega)\pi_\omega = (W, w)$. Choose any $\lambda \in \Omega$ and define $k = \phi_\lambda\pi_\lambda$. Then $(x_\omega)k = w$, so by Theorem 3 k is independent of the choice of λ. To show that k induces a homomorphism of $(Y, (x_\omega))/Q$ onto (W, w), it suffices to show that $(y, z) \in Q$ implies $yk = zk$.

Assume $(y, z) \in Q$ can be found such that $yk \neq zk$. Since W is compact Hausdorff $\mathcal{C}(W)$ separates points. Choose $f \in \mathcal{C}(W)$ so $ykf \neq zkf$. That is, $y\phi_\lambda\pi_\lambda f \neq z\phi_\lambda\pi_\lambda f$. But this is a contradiction since $(y, z) \in Q$ and $\phi_\lambda\pi_\lambda f \in \mathfrak{F}$.

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