A DIFFERENTIAL IN THE ADAMS SPECTRAL SEQUENCE

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It has been known for some time that the cohomology of the mod 2 Steenrod algebra A admits squaring operations. (For example, see [**4**].) Since the cohomology of A occurs as the E_2 term of the mod 2 Adams spectral sequence $\{E_r(S^0)\}$ [**1**], it is natural to ask if these squaring operations are in any way related to the structure of the spectral sequence. In §3 we shall prove a theorem which evaluates the differential d_2 on $\alpha\cup_1\alpha$ if α is a permanent cycle.

1. We let A denote the mod 2 Steenrod algebra and $B(A)$ the standard bar resolution [**2**, p. 32]. We let $\Delta: B(A)\to B(A)\otimes B(A)$ denote the diagonal map [**2**, p. 32] and ρ the switching map $B(A)\otimes B(A)\to B(A)\otimes B(A)$.

Δ and $\rho\Delta$ are chain homotopic. Any chain homotopy $S: \Delta\simeq\rho\Delta$ can be used to define a product $\cup_1$ in $\mathrm{Hom}_A(B(A), Z_2)$. By standard methods [**4**, p. 24] the $\cup_1$ product defines for any element $\alpha\in H^{s,r}(A)$ an element $\alpha\cup_1\alpha\in H^{2s-1,2r}(A)$. Any two chain homotopies S_1, $S_2: \rho\Delta\simeq\Delta$ will give the same value for $\alpha\cup_1\alpha$ and in particular will agree with value obtained by using the specific chain homotopy χ given on p. 36 of [**2**].

2. In dealing with the Adams spectral sequence, we shall use the formulation given in [**1**] with such additional comments as we make here. We shall use freely the definitions and notations of [**1**] in the remainder of this paper.

Our first observation is that a modification of the techniques of Lemma 1 on p. 46 of [**3**] can be used to give the following version of [**1**, Lemma 3.4]:

LEMMA (2.1). *Using the notations of* [**1**, *p.* 189], *we assume we are given a map of left A-complexes $\phi: D\to C$ covering $f^*: H^*(Z)\to H^*(X)$. Then there exists a map $g: Y_0\to W_0$ equivalent to S^nf with $g(Y_s)\subset W_s$ (for $s\leqq k$) and such that $g^*: H^*(W_s, W_{s+1})\to H^*(Y_s, W_{s+1})$ realizes ϕ.*

NOTE. In (2.1) and elsewhere we omit explicit mention of the dimension of skeletons to which the conclusions of (2.1) apply. For any given argument here, one may choose n, k and l "large enough."

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In the following lemma we let C be an acyclic resolution of $H^*(S^0)$ by free A-modules and $Y_0 \supset Y_1 \supset \cdots \supset Y_k$ a realization of C with Y_0 having the same homotopy type as S^n.

LEMMA (2.2). *Let $\gamma \in \pi_{n+q}(Y_m)$, $m+1<k$, and denote by $\bar{\gamma}$ its image in $E_2^{m,m+q}(S^0)$. Then there exists an element $\xi \in \pi_{n+q}(Y_{m+1})$ whose image in $\pi_{n+q}(Y_m)$ is 2γ and whose image in $E_2^{m+1,m+1+q}(S^0)$ is $h_0\bar{\gamma}$.*

PROOF. Let $f: S^{n+q} \to Y_m$ represent γ. Let $X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_{k-m}$ be a realization for C with X_0 having the same homotopy type as $S^{n+q} = S^{n+q}(S^0)$. Then by [1, Lemma 3.4] there exists a map $g: X_0 \to Y_m$ equivalent with f as a map into Y_m and such that $g(X_i) \subset Y_{m+i}$, $i \leqq k-m$.

Now (2.2) is clearly true for γ a generator of $\pi_0^S(S^0)$, that is, there exists $u: S^{n+q} \to X_1$ such that the image of its homotopy class in $E_2^{1,1}(S^0)$ is h_0. Then the composite $gu: S^{n+q} \to Y_{m+1}$ induces the Yoneda product representation of $h_0\bar{\gamma}$. The homotopy class of gu is the required element ξ.

3. We are now ready to prove the following result:

THEOREM. *Let $\alpha \in H^{s,s+p}(A) \approx E_r^{s,s+p}(S^0)$ be a permanent cycle in the Adams spectral sequence. Then*

(i) *$d_2(\alpha \cup_1 \alpha) = h_0\alpha^2$ if p is odd, and*

(ii) *$\alpha \cup_1 \alpha$ is a permanent cycle if p is even.*

REMARK. Part (i) could be viewed as a generalization of Theorem 1.1 of [1]. (Recall that $h_n \cup_1 h_n = h_{n+1}$.) Part (ii) is probably related to the fact that, for $\bar{\alpha} \in \pi_{2k}^S(S^0)$ and $\bar{\alpha}$ of order 2, the stable Toda bracket $\langle \bar{\alpha}, 2_\iota, \bar{\alpha} \rangle$ is divisible by 2 [7, p. 33] and the heuristic argument that $\alpha \cup_1 \alpha$ is half the Massey product $\langle \alpha, 2, \alpha \rangle$ [2, p. 47]—"heuristic" because we are working mod 2. A clarification of this is likely to require analysis along the lines of Moss's theorem, which, among other things, discusses the relation of Massey products in $H^{*,*}(A)$ to Toda brackets in $\pi_*^S(S^0)$ [6].

PROOF. Suppose n is even and large relative to p and s. Let $X_0 \supset X_1 \supset \cdots \supset X_k$, $k > s+1$, be a realization for $B(A)$ with X_0 having the homotopy type of S^n. Set

$$Y_c = \bigcup_{a+b=c} X_a \wedge X_b, \qquad (K \wedge L = K \times L / K \vee L).$$

Then Y_c is a realization of $B(A) \otimes B(A)$ with Y_0 having the homotopy type of $S^{2n} = S^n \wedge S^n$. Let $\tau: X_0 \wedge X_0 = Y_0 \to Y_0$ be the switching map. Then τ is a realization of $\rho: B(A) \otimes B(A) \to B(A) \otimes B(A)$.

Let $W_0 \supset W_1 \supset \cdots \supset W_m$, $m \geqq 2k$, be a realization of $B(A)$ with W_0 having the homotopy type of S^{2n}. By (2.1), there exists a map $\mu\colon Y_0 \to W_0$ realizing Δ. Also, since n is even, $\mu\tau$ is a realization of $\rho\Delta$.

By Lemma 3.5 of [1], there exists a homotopy $h\colon I \times Y_0 \to W_0$ such that $h_0 = \mu\tau$, $h_1 = \mu$ and $h(I \times Y_i) \subset W_{i-1}$. We may assume that the base point $y \in \bigcap_i Y_i$ and that h preserves base point. Now

$$h^*\colon \quad H^*(W_{i-1}, W_i) \to H^*(I \times Y_i, \dot{I} \times Y_i \cup I \times U_{i+1})$$

defines a chain homotopy $S\colon \Delta \simeq \rho\Delta$.

Since α is a permanent cycle, we may choose a map $u\colon S^{n+p} \to X_s$ to represent α. Denote by $\bar{u}$ its homotopy class in $\pi_{p+n}(X_s)$. It follows that the composite

$$\begin{aligned} &\theta(E^{2p+2n+1}, S^{2p+2n}) \\ &\qquad \to (I, \dot{I}) \times (S^{n+p} \wedge S^{n+p}, *) \xrightarrow{1 \times (u \wedge u)} (I, \dot{I}) \times (X_s \wedge X_s, y) \\ &\qquad \to (W_{2s-1}, W_{2s}) \end{aligned}$$

represents $\alpha \cup_1 \alpha$.

By [4, Lemma 22.3], $\partial_*\theta \in \pi_{2p+2n}(W_{2n})$ is 0 if p is even, proving part (ii); $\partial_*\theta = 2[\mu \circ (u \wedge u)]$ if p is odd. Since $\mu \circ (u \wedge u)$ represents α^2, it follows by Lemma (2.2) that there exists a map $f\colon S^{2p+2n} \to W_{2s+1}$ such that f represents $h_0\alpha^2$ and $f \simeq \theta | S^{2p+2n}$ in W_{2s}. By using a specific homotopy, one may construct an element $\bar{\theta} \in \pi_{2p+2n+1}(W_{2s-1}, W_{2s+1})$ such that $\partial_*\bar{\theta} = [f]$ and such that the image of $\bar{\theta}$ in $\pi_{2p+2n+1}(W_{2s-1}, W_{2s})$ is θ. This completes the proof.

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