A DIFFERENTIAL IN THE ADAMS SPECTRAL SEQUENCE

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It has been known for some time that the cohomology of the mod 2 Steenrod algebra $A$ admits squaring operations. (For example, see [4].) Since the cohomology of $A$ occurs as the $E_2$ term of the mod 2 Adams spectral sequence $\{E_r(S^0)\}$ [1], it is natural to ask if these squaring operations are in any way related to the structure of the spectral sequence. In §3 we shall prove a theorem which evaluates the differential $d_2$ on $\alpha \cup_1 \alpha$ if $\alpha$ is a permanent cycle.

1. We let $A$ denote the mod 2 Steenrod algebra and $B(A)$ the standard bar resolution [2, p. 32]. We let $\Delta: B(A) \to B(A) \otimes B(A)$ denote the diagonal map [2, p. 32] and $\rho$ the switching map $B(A) \otimes B(A) \to B(A) \otimes B(A)$.

$\Delta$ and $\rho \Delta$ are chain homotopic. Any chain homotopy $S: \Delta \simeq \rho \Delta$ can be used to define a product $\cup_1$ in $\text{Hom}_A(B(A), Z_2)$. By standard methods [4, p. 24] the $\cup_1$ product defines for any element $\alpha \in H^{*+r}(A)$ an element $\alpha \cup_1 \alpha \in H^{*+r}(A)$. Any two chain homotopies $S_1, S_2: \rho \Delta \simeq \Delta$ will give the same value for $\alpha \cup_1 \alpha$ and in particular will agree with value obtained by using the specific chain homotopy $\chi$ given on p. 36 of [2].

2. In dealing with the Adams spectral sequence, we shall use the formulation given in [1] with such additional comments as we make here. We shall use freely the definitions and notations of [1] in the remainder of this paper.

Our first observation is that a modification of the techniques of Lemma 1 on p. 46 of [3] can be used to give the following version of [1, Lemma 3.4]:

**Lemma (2.1).** Using the notations of [1, p. 189], we assume we are given a map of left $A$-complexes $\phi: D \to C$ covering $f^*: H^*(Z) \to H^*(X)$. Then there exists a map $g: Y_0 \to W_0$ equivalent to $Sf$ with $g(Y_s) \subseteq W_s$ (for $s \leq k$) and such that $g^*: H^*(W_n, W_{n+1}) \to H^*(Y_s, W_{s+1})$ realizes $\phi$.

**Note.** In (2.1) and elsewhere we omit explicit mention of the dimension of skeletons to which the conclusions of (2.1) apply. For any given argument here, one may choose $n, k$ and $l$ “large enough.”

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In the following lemma we let $C$ be an acyclic resolution of $H^*(S^0)$ by free $A$-modules and $Y_0 \supset Y_1 \supset \cdots \supset Y_k$ a realization of $C$ with $Y_0$ having the same homotopy type as $S^n$.

**Lemma (2.2).** Let $\gamma \in \pi_{n+q}(Y_m)$, $m+1 < k$, and denote by $\widetilde{\gamma}$ its image in $E^{e,m+q}_2(S^0)$. Then there exists an element $\xi \in \pi_{n+q}(Y_{m+1})$ whose image in $\pi_{n+q}(Y_m)$ is $2\gamma$ and whose image in $E^{e+1,m+1+q}_2(S^0)$ is $h_0\widetilde{\gamma}$.

**Proof.** Let $f: S^{n+q} \to Y_m$ represent $\gamma$. Let $X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_{k-m}$ be a realization for $C$ with $X_0$ having the same homotopy type as $S^{n+q} = S^{n+q}(S^0)$. Then by [1, Lemma 3.4] there exists a map $g: X_0 \to Y_m$ equivalent with $f$ as a map into $Y_m$ and such that $g(X_i) \subseteq Y_{m+i}$, $i \leq k-m$.

Now (2.2) is clearly true for $\gamma$ a generator of $\pi^S_0(S^0)$, that is, there exists $u: S^{n+q} \to X_1$ such that the image of its homotopy class in $E^{1,1}_2(S^0)$ is $h_0$. Then the composite $gu: S^{n+q} \to Y_{m+1}$ induces the Yoneda product representation of $h_0\widetilde{\gamma}$. The homotopy class of $gu$ is the required element $\xi$.

3. We are now ready to prove the following result:

**Theorem.** Let $\alpha \in H^{s+1}(A) \otimes H^{s+1}(S^0)$ be a permanent cycle in the Adams spectral sequence. Then

(i) $d_2(\alpha \cup \alpha) = h_0\alpha^2$ if $p$ is odd, and

(ii) $\alpha \cup \alpha$ is a permanent cycle if $p$ is even.

**Remark.** Part (i) could be viewed as a generalization of Theorem 1.1 of [1]. (Recall that $h_n \cup_1 h_n = h_{n+1}.$) Part (ii) is probably related to the fact that, for $\alpha \in \pi^S_2(S^0)$ and $\bar{\alpha}$ of order 2, the stable Toda bracket $\langle \bar{\alpha}, 2, \bar{\alpha} \rangle$ is divisible by 2 [7, p. 33] and the heuristic argument that $\alpha \cup_1 \alpha$ is half the Massey product $\langle \alpha, 2, \alpha \rangle$ [2, p. 47]—"heuristic" because we are working mod 2. A clarification of this is likely to require analysis along the lines of Moss' theorem, which, among other things, discusses the relation of Massey products in $H^*(A)$ to Toda brackets in $\pi^S_*(S^0)$ [6].

**Proof.** Suppose $n$ is even and large relative to $p$ and $s$. Let $X_0 \supset X_1 \supset \cdots \supset X_{k-b}$, $k > s+1$, be a realization for $B(A)$ with $X_0$ having the homotopy type of $S^n$. Set

$$Y_s = \bigcup_{a+b=c} X_a \wedge X_b, \quad (K \wedge L = K \times L/K \vee L).$$

Then $Y_s$ is a realization of $B(A) \otimes B(A)$ with $Y_0$ having the homotopy type of $S^{2n} = S^n \wedge S^n$. Let $\tau: X_0 \wedge X_0 = Y_0 \to Y_0$ be the switching map. Then $\tau$ is a realization of $\rho: B(A) \otimes B(A) \to B(A) \otimes B(A)$. 

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Let $W_0 \supset W_1 \supset \cdots \supset W_m$, $m \geq 2k$, be a realization of $B(A)$ with $W_0$ having the homotopy type of $S^{2n}$. By (2.1), there exists a map $\mu: Y_0 \to W_0$ realizing $\Delta$. Also, since $n$ is even, $\mu_\ast$ is a realization of $\rho_\Delta$.

By Lemma 3.5 of [1], there exists a homotopy $h: I \times Y \to W_0$ such that $h_0 = \mu_\ast$, $h_1 = \mu$ and $h(I \times Y_i) \subset W_{i-1}$. We may assume that the base point $y \in \bigcap_i Y_i$ and that $h$ preserves base point. Now

$$h^*: H^*(W_{i-1}, W_i) \to H^*(I \times Y_i, I \times Y_i \cup I \times U_{i+1})$$

defines a chain homotopy $S: \Delta \simeq \rho_\Delta$.

Since $\alpha$ is a permanent cycle, we may choose a map $u: S^{n+p} \to X_s$ to represent $\alpha$. Denote by $\bar{u}$ its homotopy class in $\pi_{n+p}(X_s)$. It follows that the composite

$$\partial(\eta^{2p+2n+1}S^{2p+2n})$$

\[\to (I, I) \times (S^{n+p} \wedge S^{n+p}, \ast) \xrightarrow{1 \times (u \wedge u)} (I, I) \times (X_s \wedge X_s, y) \]

\[\to (W_{2s-1}, W_{2s})\]

represents $\alpha \cup_\ast \alpha$.

By [4, Lemma 22.3], $\partial_s \theta \in \pi_{2p+2n}(W_{2n})$ is 0 if $p$ is even, proving part (ii); $\partial_s \theta = 2[\mu \circ (u \wedge u)]$ if $p$ is odd. Since $\mu \circ (u \wedge u)$ represents $\alpha^2$, it follows by Lemma (2.2) that there exists a map $f: S^{2p+2n} \to W_{2s+1}$ such that $f$ represents $h_\ast \alpha^2$ and $f \sim \theta$ $S^{2p+2n}$ in $W_{2s}$. By using a specific homotopy, one may construct an element $\bar{\theta} \in \pi_{2p+2n+1}(W_{2s-1}, W_{2s+1})$ such that $\partial_s \bar{\theta} = [f]$ and such that the image of $\bar{\theta}$ in $\pi_{2p+2n+1}(W_{2s-1}, W_{2s})$ is $\theta$. This completes the proof.

**Bibliography**


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