A THEOREM ON INFINITE POSITIVE MATRICES[1]

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1. Let $A=(a_{ij})$ be an infinite matrix with positive elements $a_{ij}>0, i, j=0, 1, \cdots$, (matrices (a_{ij}) with $a_{ij}>0$ will be called in the sequel positive matrices).

It was proved in [**3**], that

(1) if A is a *finite* positive matrix, a unique doubly stochastic matrix T exists such that $T=D_1AD_2$ where D_1 and D_2 are diagonal matrices with all elements on the diagonal positive and are unique up to a scalar factor.

The method used in [**3**], and introduced first in [**4**], is a constructive one and consists in alternate normalizing rows and columns of A and proving the convergence of this procedure. Another proof of (1) was given in [**1**]. This second proof uses besides Brouwer's fixed point theorem the fact, that

(2) the set $\{x=(x_0, x_1, \cdots, x_n);\ x_i$ real numbers, $\sum_{i=0}^n x_i^2=1$ and $x_i\geqq 0\}$ is homeomorphic to an n-dimensional ball.

Although a purely existential one, this second proof contains a statement about the existence of directions of fixed points for some mapping defined by help of a *finite* matrix A. In this paper we note that statement (1) does not hold for *infinite* matrices and prove a theorem generalizing properly (1) to the case of infinite matrices. Essentially, both proofs in [**1**] and in [**3**] could be, with some nontrivial changes, applied to give the desired generalization. The difficulty in generalizing the proof given in [**3**] consists i.a. in the fact that for an infinite matrix $\sum_j a_{ij}$ (or $\sum_i a_{ij}$) is not always finite. The idea of our proof is similar to that of [**1**] except that (2) is not used and that Brouwer's theorem is replaced by the theorem of Schauder (see [**2**]). In the sequel a matrix $A=(a_{ij})$ with $a_{ij}>0$ will be called a positive matrix and a diagonal matrix with positive diagonal elements will be called a positive diagonal matrix. Finally $\delta_{ij}=0,\ i\neq j;\ 1,\ i=j$, will denote the delta of Kronecker.

2. Before generalizing (1) to the case of infinite matrices let us note that

(3) if $A=(a_{ij})$ is infinite with $a_{ij}=1$ for $i, j=0, 1, 2, \cdots$ then positive diagonal matrices D_1 and D_2 for which $T=D_1AD_2$ is doubly stochastic do not exist.

Received by the editors September 11, 1967.

[1] This research was supported by the U. S. Air Force Grant AF-AFOSR 396-63, monitored by the Office of Scientific Research.

Indeed, if $D_1=(\delta_{ij}p_i)$ and

$$D_2 = (\delta_{ij}q_j), \qquad p_i > 0, \qquad q_j > 0, \qquad i,j = 0, 1, 2, \cdots$$

then for a doubly stochastic matrix $T=D_1AD_2$ one has $t_{ij}=p_iq_j$ and $p_i=p_j$ for all $i, j=0, 1, \cdots$, which is impossible.

We show now, that

(4) if $A=(a_{ij})$ with $a_{ij}=1$ for all $i, j=0, 1, \cdots$ and if there exist positive diagonal matrices $D_1=(\delta_{ij}p_i)$ and $D_2=(\delta_{ij}q_j)$ such that for $T=D_1AD_2=(t_{ij})$ one has $\sum_j t_{ij}=\alpha_i$ and $\sum_i t_{ij}=\beta_j$, $i, j=0, 1, \cdots$, then $\sum_{i=0}^\infty \alpha_i=\sum_0^\infty \beta_j<\infty$.

Indeed, condition $\sum_{j=0}^\infty t_{0j}=\alpha_0$ i.e. $p_0\sum_0^\infty q_j=\alpha_0$ implies $\sum_0^\infty q_j<\infty$ and similarly $\sum_0^\infty p_i<\infty$. But then$(\sum_0^\infty p_i)(\sum_0^\infty q_j)=\sum_0^\infty \alpha_i=\sum_0^\infty \beta_j<\infty$.

Property (4) justifies the assumption $\sum_0^\infty \alpha_i=\sum_0^\infty \beta_j<\infty$ made in the following

THEOREM. *Let $A=(a_{ij})_{i,j=0,1,\cdots}$ be an infinite positive matrix such that*

(a) *there exists a constant M with $a_{ij}\leqq M$ and*

(b) *there exists a column (say the 0th column) and constants L_0 and M_0 such that for every $i, k=0, 1, \cdots$ one has $a_{i0}\leqq M_0a_{ik}$ and $a_{ik}\leqq L_0a_{i0}$. Let further $\{\alpha_i\}$ and $\{\beta_j\}$ be sequences of positive numbers such that $\sum_0^\infty \alpha_i=\sum_0^\infty \beta_j<\infty$.*

Then there exist positive diagonal matrices D_1 and D_2 such that for $T=D_1AD_2=(t_{ij})$ one has

(c) $\sum_j t_{ij}=\alpha_i$ *and* $\sum_i t_{ij}=\beta_j$, $i, j=0, 1, \cdots$.

PROOF. Putting $N_0=1$ and $N_i=N\geqq 1$ for $i\geqq 1$ and multiplying A on the right by the matrix $D=(\delta_{ij}N_i)$ one can by choosing N sufficiently large obtain by (b) that, for the matrix $B=AD=(b_{ij})$,

(d) $b_{i0}\leqq b_{ik}$ and $b_{i0}\leqq(\beta_0/\sum_{k\geq 1}\beta_k)b_{ik}$ holds for every $i, k=0, 1, \cdots, k\neq 0$.

It suffices obviously to find positive diagonal matrices P and Q such that $T=PBQ=(t_{ij})$ satisfies (c) (then $D_1=P$ and $D_2=DQ$). Now consider the equations

(e_1) $u_i\sum_j b_{ij}x_j'=\alpha_i$,

(e_2) $x_k\sum_i b_{ik}u_i=\beta_k$, $i, k=0, 1, \cdots$.

Expressing x_k in terms of x_j' we get

(f) $x_k=\beta_k/f_k(\{x_j'\})$, where $f_k(\{x_j'\})=\sum_{i\geq 0}(\alpha_i b_{ik}/\sum_{j\geq 0} b_{ij}x_j')$.

Evidently, if one finds a sequence $\{x_j'\}$ with $x_j'>0$ such that in (f) $x_k=x_k'$ for every $k\geqq 0$, then calculating u_i from (e_1) and putting $P=(\delta_{ij}u_i)$ and $Q=(\delta_{ij}x_j)$ we have the desired matrices P and Q. In other words one looks for any fixed point $x=\{x_k\}_{k=0,1,\cdots}$, $x_k>0$ of the mapping defined by (f). To get such a fixed point let us denote

$\xi_k = x_k'/x_0'$ and $\eta_k = x_k/x_0$, $k=1, 2, \cdots$, (we call $\{\xi_k\}$ and $\{\eta_k\}$ "directions").

Then by (f) one has

$$\text{(g)} \qquad \eta_k = \frac{\beta_k}{\beta_0} \frac{g_0(\{\xi_j\})}{g_k(\{\xi_j\})}, \quad \text{where} \quad g_k(\{\xi_j\}) = \sum_{i \geq 0} \frac{\alpha_i b_{ik}}{b_{i0} + \sum_{j \geq 1} b_{ij}\xi_j}.$$

Let us confine ourselves to $\xi_j \geqq 0$ such that $\sum_{j\geq 1} \xi_j \leqq 1$, i.e. such that the point $x = (\xi_1, \xi_2, \cdots)$ belongs to the intersection $C \cap S$ of the cone $C = \{x = (\xi_1, \xi_2, \cdots); \xi_i \geqq 0, x \in l\}$ in the Banach space l with the unit ball S of this space.[2] Since $\sum \alpha_i < \infty$ we obtain by (a) that η_k exists for all $k = 1, 2, \cdots$ and obviously by (b) $\eta_k > 0$. By (d) it follows that $\eta_k \leqq \beta_k/\beta_0$ and that $\sum_{k=1}^{\infty} \eta_k \leqq 1$.

Thus, by $\sum \beta_k < \infty$, formula (g) defines a continuous mapping F of $C \cap S$ into a compact subset of $C \cap S$. By the fixed point theorem of Schauder (see [2]) there exists a point $\bar{x} = (\bar{\xi}_1, \bar{\xi}_2, \cdots)$ such that $F(\bar{x}) = \bar{x}$. This point is an invariant direction of the mapping F. Take as in [1] any point $(x_0', x_1', \cdots)$ on this direction with $x_0' > 0$. Then $x_j = \theta x_j'$, $j = 0, 1, \cdots$, and putting x_j' into (e_1) we obtain the sequence $\{u_i\}_{i=0,1,\ldots}$. Then by (e_1) and (e_2) we have

$$\sum_i u_i \sum_j b_{ij} x_j' = \sum \alpha_i = \sum \beta_i = \theta \sum u_i \sum_j b_{ij} x_j'.$$

Thus $\theta = 1$ and the sequences $\{u_i\}$ and $\{x_j\}$ satisfy both (e_1) and (e_2).

The theorem is proved.

REMARKS. Let us note that in case of a finite positive matrix $A = (a_{ij})$ all the assumptions of the Theorem hold. Finally let us note that if $A = (a_{ij})_{i,j=0,1,\ldots}$ is infinite and $a_{ij} = 1/2^{j+1}$, $i, j = 0, 1, 2, \cdots$, then A is obviously a stochastic matrix but the argument applied in (3) shows that positive diagonal matrices D_1 and D_2 for which $T = D_1 A D_2$ is doubly stochastic do not exist.

References

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[2] l denotes the Banach space of all sequences $x = (\xi_1, \xi_2, \cdots)$ with ξ_i real and $\sum |\xi_i| < \infty$.