RIGIDITY OF GENERALIZED UNISERIAL AND FROBENIUS ALGEBRAS

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Introduction. In [2], [4] Gerstenhaber, Nijenhuis and Richardson introduced the concept of deformations of an algebra A over a field. In this note we consider the question of whether there are any reasonable classes of algebras which admit nontrivial deformations but for which all the deformations remain in the class. The main theorem states that if A is a generalized uniserial basic algebra, then every deformation of A is generalized uniserial. The theorem actually gives a complete description of all deformations of a generalized uniserial basic algebra.

The second theorem states that the class of Frobenius algebras is a class closed under deformations.

We begin by setting up the notation. Throughout A will denote an associative algebra over a field k which admits a Wedderburn decomposition, $A=S+N$ where S is K separable and N is the Jacobson radical. A will be called a basic algebra if the simple components of A are one dimensional over k and in this case we will write $S=\sum_{i=1}^{n} ke_i$ where e_i is the identity of the ith component of S.

A generalized uniserial algebra A is an algebra such that for any primitive idempotent e the left (resp. right) modules Ae (eA) have a unique decomposition series.

Following [2], $A_0=A\otimes_k k((t))$ where $k((t))$ is the field of quotients of the power series ring over k in one indeterminant t. By a deformation of A we will mean an associative multiplication induced by a bilinear function $f_t\colon A\otimes_k A\to A_0$ of the form

$$f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) \cdots .$$

Call such functions f_t multiplicative functions. Two multiplicative functions f_t and g_t are said to be equivalent, $f_t\sim g_t$, if there exists a linear function $\psi_t\colon A\to A_0$ of the form $\psi_t(a)=a+\varphi_1(a)t+\varphi_2(a)t^2+\cdots$ such that $f_t(a, b)=\psi_t^{-1}(g_t(\psi_t(a), \psi_t(b)))$. It is known, see [5], [7], that every deformation given by a multiplicative function $f_t=ab+tF_1(a, b)\cdots$ is equivalent to a deformation given by a multiplicative function $g_t(a, b)=ab+tG_1(a, b)+t^2G_2(a, b)\cdots$ where $G_i(a, S)=G_i(S, a)=0$ for all $s\in S$ and $a\in A$.

The first theorem we need is due to Kupisch [3].

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THEOREM. *Let A be a generalized uniserial basic algebra. Then $A=S+M^{(1)}+M^{(2)}+\cdots+M^{(e)}$ (K-direct) where $M^{(1)}+N^2=N$ and $M^{(i)}+N^{i+1}=N^i$ and A has a basis $\{b_i\}_{i=1}^n$ such that $b_1=e_1, \cdots, b_{p_0}=e_{p_0}$, $b_{p_0+1}\in N^{(1)} \cdots b_{p_1}\in N^{(1)}$, $b_{p_1+1}\cdots b_{p_2}\in N^{(2)}\cdots b_{p_{e-1}}\cdots b_n\in N^{(e)}$ and the nonzero products of the $b_{p_i+1}\cdots b_{p_i}$, with the $b_{p_0+1}\cdots b_{p_1}$ are the $b_{p_{i+1}+1}\cdots b_{p_{i+2}}$ and for each b_α there exists a unique i such that $e_ib_\alpha=b_\alpha$ and a unique j such that $b_\alpha e_j=b_\alpha$. We will write $e_i m e_j$ for the $b_{p_0+1}\cdots b_{p_2}$.*

We will need also the following lemma the proof of which is routine and will be omitted.

LEMMA. *Let K be any field and let $A=K[x]/I$, I any ideal of $K[x]$. Then A is generalized uniserial.*

We now prove two lemmas the first of which is a special case of the desired result and the second a reduction theorem.

LEMMA. *The only deformations of $L[x]/(x^n)$ are of the form $L[x]/P(x)$ where $P(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}+x^n$ and $L=K((t))$.*

PROOF. Since $F_1\in H^2(A, A)$ we claim $F_1\sim G_1$ where $G_1(x, x^j)=0$ for $j<n-1$. This follows by setting

$$\phi(1) = 0, \qquad \phi(x^j) = F_1(x, x^j) + x\phi(x^{j-1})$$

and letting $G_1=F_1-\delta\phi$.

Notice now if we set $g_t(a, b)=ab+tG_1(a, b)$ we have

$$A_{g_t} \cong L[x]/(x^n + tG_1(x, x^{n-1})).$$

Thus $G_1(a, G_1(b, c)) - G_1(G_1(a, b), c) = 0$, $a, b, c \in K[x]/(x^n)$.

Now let f_t be a multiplicative function of the form

$$f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) \cdots$$

then by the above $f_t\sim f_t'$ where

$$f_t'(a, b) = ab + tG_1(a, b) + t^2G_2(a, b) \cdots$$

and $G_1(x, x^j)=0, j<n-1$, and $G(a, G(b, c))-G(G(a, b), c)=0$ for all a, b, c, in $K[x]/(x^n)$. By this last equation we see that $\delta G_2=0$, so we can repeat the process to G_2, so that $f_t\sim f_t''$ where

$$f_t''(a, b) = ab + tG_1(a, b) + t^2G_2'(a, b) + \cdots$$

where $G_2'(x, x^j)=0, j<n-1$; and it follows that $G_3'=0$. Continuing by induction we find $f_t\sim h_t$ where

$$h_t(a, b) = ab + tH_1(a, b) + t^2H_2(a, b) \cdots$$

and $H_i(x, x^j)=0, j<n-1$. Now set

$$p(x) = x^n + tH_1(x, x^{n-1}) + t^2H_2(x, x^{n-1}) \cdots .$$

One can easily verify that $A_{h_t} \cong L[x]/(p(x))$.

LEMMA. *Let A be a generalized uniserial basic algebra where $N = M+M^2+M^3+ \cdots +M^n$ as in Theorem 3.9. Then if $F \in H^2(A, A)$ and $F(M^i, M^j) \in \sum_{l \geq i+j+1} M^l$, F is trivial.*

PROOF. Let $e_ime_j = {}_im_j$ be the basis of M from Kupisch's Theorem. On the basis of M^2 define

$$\begin{aligned} \phi_1(m) &= -F({}_in_j, {}_in_k) \qquad \text{for } m = {}_im_j\, {}_jm_k, \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and extend ϕ_1 linearly to all of A in the obvious way. Now $G_1 = F+\delta\phi_1$ has the property that $G_1(M, M)=0$. Define ϕ_2 on the basis of M^3 by

$$\begin{aligned} \phi_2(m) &= -G({}_im_j\, {}_jm_k, {}_km_p), \qquad m = {}_in_j\, {}_jm_k\, {}_km_p, \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

and extend to all of A.

Setting $G_2=G_1+\delta\phi_2$, G_2 has the property that $G_2(M^2, M)=0$. But since $\delta G_2=0$, $G_2(M, M^2)=0$. Continuing inductively we obtain $F\sim G$ where $G(M, N)=0$. It follows that $G=0$ since $\delta G=0$.

THEOREM. *If A is a generalized uniserial basic algebra, then all deformations of A_0 are generalized uniserial.*

PROOF. Let F_t be a multiplicative function. We can assume that f_t has the form $f_t(a, b)=ab+tF_1(a, b)+t^2F_2(a, b) \cdots$ and F_1 is not zero in $H^2(A, A)$. Let $F=F_1+F_2$ where

$$F_1(M^i, M^j) \in \sum_{k \leq i+j} M^k, \qquad F_2(M^i, M^j) \in \sum_{k \geq i+j+1} M^k.$$

One checks easily that F_1 and $F_2 \in H^2(A, A)$, and therefore we can assume $F_2=0$ by the above.

Now since F is nontrivial it follows that $F(M, N)\neq 0$; so there exists ${}_im_j$ such that $F({}_im_j, M^l)\neq 0$. Pick ${}_im_j$ so l is minimal with this property. Now for some ${}_tm_k\, {}_km \cdots {}_bm_p \in M$ we have

$$F({}_im_j, {}_jm_k\, {}_km \cdots {}_bm_p) = {}_in_j\, {}_jn_p \neq 0,$$

where ${}_jy_p \in \sum_{1 \leq k \leq -1} M^k$. Set

$$\begin{aligned} \phi(z) &= {}_jy_p \qquad z = {}_jm_k{}_km \cdots {}_bm_p, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

extending to all of A. Form $F-\delta\phi=G$. G has the property that $G(jm_k, km \cdots {}_bm_p)\neq 0$ and note that ${}_km_q \cdots {}_bm_p \in M^{l-1}$. Repeating the process to G we find $F\sim G_1$ where $G_1({}_im_j, {}_jm_p)\neq 0$. But then

$$G_1({}_im_j, {}_jm_p) = \alpha_jm_p \qquad \text{some } \alpha \in K,$$

but by the uniqueness of the idempotents on the left we have $j=i$, which implies $j=p$, which implies $i=p$. So in the above we really had

$$G_1({}_pm_p, {}_pm_p) = \alpha_pm_p.$$

This implies that Ae_p is a ring direct summand of A and each $F_i(Ae_p, Ae_p) \subset Ae_p$. Now $Ae_p=e_pAe_p\cong K[x]/(x^n)$ some n; so it follows that we are only deforming ring direct summands which are of the form $K[x]/(x^n)$, and they remain ring direct summands. The result follows.

We now drop all assumptions on A except that A be finite dimensional over K.

THEOREM. *Let A be a Frobenius algebra over k. Then every deformation of A is Frobenius as an algebra over $K((t))$.*

PROOF. Since A is Frobenius there exists a linear functional $\lambda: A\to K$ such that there are no ideals (right or left) in $\ker\lambda$ [1, p. 414]. $A\otimes k((t))$ is Frobenius and in fact if we define $\tilde{\lambda}(a\otimes p(t)) =\lambda(a)p(t)$ kernel $\tilde{\lambda}$ contains no ideals in $A\otimes k((t))$. Now let f_t be a deformation of A. $\tilde{\lambda}$ is still a linear functional on the deformed algebra and one easily checks that kernel $\tilde{\lambda}$ contains no ideals. Thus A_{f_t} is Frobenius.

For further theorems on properties perserved under deformations see [5], [6].

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