A PROBLEM OF WIENER AND THE FAILURE OF A PRINCIPLE FOR FOURIER SERIES WITH POSITIVE COEFFICIENTS

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Wiener proved the following theorem [1].

THEOREM A. *Let*

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

be analytic in $|z| < 1$, *and assume* $a_n \geqq 0$. *Furthermore, assume for some* $\delta > 0$,

$$\sup_{0 \leqq r < 1} \int_{-\delta}^{\delta} |F(re^{i\theta})|^2 d\theta \leqq M < \infty.$$

Then

$$\sup_{0 \leqq r < 1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^2 d\theta \leqq N < \infty.$$

Wiener then asked if the analogue of his theorem were true for H^1 [1]. (That is, whether the square in the integrals of Theorem A may be replaced by the first power.) This question has added interest now because Boas [2], [3] has shown that if a function with positive Fourier series coefficients is "good" in a neighborhood of zero, then the function is "good" everywhere for several senses of the word "good." In this note we point out that a result of Hardy implies a negative answer to Wiener's question, and at the same time gives an example of a trigonometric series with positive coefficients whose behaviour is good near the origin but not everywhere.

THEOREM B. *Let* $0 < p < 2$. *Then there exists*

$$F(z) = \sum_{n=1}^{\infty} a_n z^n$$

analytic in $|z| < 1$ *with* $a_n \geqq 0$ *and* $\delta > 0$ *such that*

$$\sup_{r \leqq 1} \int_{-\delta}^{\delta} |F(re^{i\theta})|^p d\theta \leqq M < \infty; \tag{1}$$

Received by the editors October 2, 1967.

[1] Supported in part by NSF Grant GP-6764.

$$\sup_{r<1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta = \infty. \tag{2}$$

REMARK. δ may be chosen to be any number less than π.

PROOF OF THEOREM B. We take $a_n = n^{-\gamma}(2+\cos\lambda\, n \cos n^{\alpha})$, with $\pi > \lambda > \delta$. Let

$$h(r, \theta) = \sum_{n=1}^{\infty} r^n n^{-\gamma} e^{in\theta}.$$

As $\theta \to 0$, $h(\theta) \simeq k_\gamma \theta^{-1+\gamma}$ with $k_\gamma \neq 0$, where $h(\theta) = \lim_{r\to 1} h(r, \theta)$. Furthermore as one would then expect

$$\sup_{0\leqq r<1} \int_{-\pi}^{\pi} |h(r, \theta)|^p d\theta \leqq M < \infty, \qquad \text{for } p(1-\gamma) < 1.$$

See [**5**, Chapter 5]. Let

$$g(r, \theta) = \sum_{n=1}^{\infty} r^n n^{-\gamma} \exp(in^{\alpha}) \exp(in\theta), \qquad 0 < \alpha < 1.$$

Hardy showed [**4**] $g(\theta) = \lim_{r\to 1} g(r, \theta)$ exists for $\theta \neq 0$, and

$$|g(\theta)| \simeq k(\alpha, \gamma) |\theta|^{-(1-\gamma-\alpha/2)/(1-\alpha)}$$

with $k(\alpha, \gamma) \neq 0$, as θ tends to zero from the left, is bounded as θ tends to zero from the right, and is otherwise continuous in $[-\pi, \pi]$. Also, $g(r, \theta)$ is bounded in any portion of the unit circle lying above a line $y = \beta(x-1)$. (x and y are here the usual rectangular coordinates and $-\infty < \beta < \infty$. The bound of course depends on β.) $F(re^{i\theta})$ is made up of two parts, one having a singularity like $h(r, \theta)$ at $r=1$, $\theta=0$; the other having singularities like that of $g(r, \theta)$ except translated to $\theta = +\lambda$ and $\theta = -\lambda$.

We choose any γ with $1/2 > \gamma > 1-1/p$. Then $\|h(r, \theta)\|_p$ will be bounded uniformly in $r<1$ and $F(re^{i\theta})$ will satisfy the hypothesis (1). But now, by choosing α sufficiently close to 1, we can make $|g(\theta)| \simeq c\theta^{-l}$ for arbitrarily large l ($c \neq 0$). Thus $F(\theta)$ is not in L^p. Hence

$$\sup_{r<1} \int_{-\pi}^{\pi} |F(re^{i\theta})|^p d\theta = \infty.$$

See [**5**, Chapter 7].

REMARK. The analogue of Wiener's Theorem holds for $p=2k$, $k=1, 2, \cdots$. (That is, Theorem B is false for these values. This follows from Wiener's Theorem. What happens for arbitrary $p>2$ seems an interesting problem.)

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