THE APPROXIMATE DIVERGENCE OPERATOR

VICTOR L. SHAPIRO[1]

1. We shall operate in Euclidean k-space, $k \geqq 2$, and shall denote by $B(x, r)$ the k-ball with center x and radius r. Similarly, by $S(x, r)$ we shall denote the $(k-1)$-sphere with center x and radius r.

If $v(x) = [v_1(x), \cdots, v_k(x)]$ is a Lebesgue measurable vector field defined almost everywhere in $B(x_0, r_0)$ (that is each component is defined almost everywhere in $B(x_0, r_0)$ and is Lebesgue measurable), we shall say that the approximate divergence of v exists at the point x_0 and equals α providing the following two facts hold:

(i) There exists a 1-dimensional Lebesgue measurable set Q situated on the positive real axis which has 0 as a point of density from the right such that for r in $(0, r_0) \cap Q$

$$\int_{S(x_0,r)} | v_j(x) n_j(x) | \, dS(x) < \infty, \qquad j = 1, \cdots, k,$$

where $n(x) = [n_1(x), \cdots, n_k(x)]$ is the outward pointing unit normal to $S(x_0, r)$ at the point x and $dS(x)$ is the natural $(k-1)$-dimensional volume element on $S(x_0, r)$.

(ii) As $r \to 0$ through the points of Q,

$$| B(x_0, r) |^{-1} \int_{S(x_0,r)} v(x) \cdot n(x) dS(x) \to \alpha$$

where $|B(x_0, r)|$ stands for the k-volume of $B(x_0, r)$.

If (i) and (ii) hold, we shall henceforth write ap div $v(x_0) = \alpha$.

It is clear that if $v(x)$ is in class C^1 in $B(x_0, r_0)$, then ap div $v(x_0)$ exists and equals the usual divergence of v evaluated at x_0.

With an integral lattice point designated by m where $m = (m_1, \cdots, m_k)$ and using the notation $(m, x) = m_1 x_1 + \cdots + m_k x_k$ and $|m| = (m, m)^{1/2}$, we shall say the k-dimensional trigonometric series $\sum a_m e^{i(m,x)}$ is in class D if for each j there is a function $v_j(x)$ in $L^1(T_k)$ which has

$$\sum{}' - i a_m m_j | m |^{-2} e^{i(m,x)}$$

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as its Fourier series. If this is the case, we shall write

$$v_j(x) \sim \sum{}' - i a_m m_j |m|^{-2} e^{i(m,x)}, \qquad j = 1, \cdots, k, \tag{1}$$

and set

$$v(x) = [v_1(x), \cdots, v_k(x)]. \tag{2}$$

(T_k, as usual, designates the k-dimensional torus, and $\sum'$ the fact that the 0-lattice point is not to be considered. For multiple trigonometric series, we shall use the notation of [1].)

We shall say that $\sum a_m e^{i(m,x)}$ is spherically convergent at the point x_0 to α if

$$\lim_{u\to\infty} \sum_{|m|\leq u} a_m e^{i(m,x_0)} = \alpha.$$

In this paper, we shall establish the following theorem:

THEOREM. *Let* $\sum a_m e^{i(m,x)}$ *be a k-dimensional trigonometric series in class D which is spherically convergent at the point* x_0 *to* α *(of finite modulus). Then*

$$\text{ap div } v(x_0) = \alpha - a_0,$$

where $v(x)$ *is the vector field defined by* (2).

A close look at the definitions given will show that the above theorem is a generalization (and an improvement) of the theorem given in [2, p. 324].

2. We now proceed with the proof of the theorem. Without loss of generality, we assume that x_0 is the origin and that both $a_0=0$ and $\alpha=0$.

For each j, we are given $v_j(x)$ in $L^1(T_k)$ satisfying (1) .We define the 1-dimensional set P to be

$$P = \left\{ r: \int_{S(0,r)} |v_j(x)|\, dS(x) < \infty \text{ for } j = 1, \cdots, k \text{ and } 0 < r < 1 \right\}.$$

Since for each j, $v_j(x)$ is also in L^1 on $B(0, 1)$, it follows from Fubini's theorem that P is a Lebesgue measurable set and furthermore that

(3) $\mu(P)=1$ where μ is 1-dimensional Lebesgue measure.

Next we set $A_u(x) = \sum_{|m|\leq u} a_m e^{i(m,x)}$. Observing that $d[J_p(t)t^{-p}]/dt = -J_{p+1}(t)t^{-p}$, where $J_p(t)$ is the Bessel function of the first kind and order p, and recalling that $a_0=0$, we obtain that for $r>0$

$$\sum_{|m|\leq u} a_m J_{k/2}(|m|r)/(|m|r)^{k/2}$$

$$= r\int_0^u A_t(0)J_{(k+2)/2}(tr)(tr)^{-k/2}dt + A_u(0)J_{k/2}(ur)(ur)^{-k/2}.$$

Since both $J_{k/2}(u)$ and $J_{(k+2)/2}(u)$ are $O(u^{-1/2})$ as $u\rightarrow\infty$ and since by assumption $A_u(0)=o(1)$ as $u\rightarrow\infty$, we conclude that

(4)
$$\lim_{u\rightarrow\infty} \sum_{|m|\leq u} a_m J_{k/2}(|m|r)(|m|r)^{-k/2} = r\int_0^\infty A_u(0)J_{(k+2)/2}(ur)(ur)^{-k/2}du.$$

The integral on the right side of the equality in (4) is absolutely convergent for every $r>0$; consequently, the limit on the left side of the equality is finite for every $r>0$. We obtain, therefore, that

(5)
$$\lim_{t\rightarrow 0} \sum_{m} a_m J_{k/2}(|m|r)(|m|r)^{-k/2}e^{-|m|t} = \lim_{u\rightarrow\infty} \sum_{|m|\leq u} a_m J_{k/2}(|m|r)(|m|r)^{-k/2} \qquad \text{for } r>0.$$

Next, we set for $t>0$

(6)
$$v_j(x, t) = \sum_m{}' - ia_m m_j |m|^{-2} e^{i(m,x)-|m|t}$$

and

(7)
$$v(x, t) = [v_1(x, t), \cdots, v_k(x, t)].$$

From a familiar computation involving Bessel functions, we obtain

(8)
$$|B(0, r)|^{-1}\int_{S(0,r)} v(x, t)\cdot n(x)dS(x) = \gamma_k \sum_m a_m J_{k/2}(|m|r)(|m|r)^{-k/2}e^{-|m|t},$$

where γ_k is a positive constant depending on k.

From (4), (5), and (8), it follows that, for $0<r<1$, $g(r)$ exists and is finite where $g(r)$ is defined by the following limit

(9)
$$g(r) = \lim_{t\rightarrow 0}\int_{S(0,r)} v(x, t)\cdot n(x)dS(x).$$

We note that for $t>0$, $v(x, t)$ is a continuous vector-valued function in x; so $g(r)$ is a Borel measurable function of r on the interval (0, 1). We recall that $v_j(x)$ is a function in $L^1(T_k)$ satisfying (1). It follows therefore from (6) and well-known facts concerning multiple Fourier series, e.g. [1, (17), p. 56], that

$$(10)\qquad \int_{T_k} | v_j(x) - v_j(x, t) | \, dx \to 0 \qquad \text{as } t \to 0 \qquad \text{for } j = 1, \cdots, k.$$

Using these facts, we next show

$$(11)\qquad \int_0^1 \left| \int_{S(0,r)} v(x) \cdot n(x) dS(x) - g(r) \right| dr = 0.$$

We do this by observing that, by Fatou's lemma and (9), the expression on the left side of (11) is majorized by

$$\liminf_{t\to 0} \int_0^1 \left| \int_{S(0,r)} [v(x) - v(x, t)] \cdot n(x) dS(x) \right| dr.$$

But this expression in turn is majorized by

$$\liminf_{t\to 0} \sum_{j=1}^{k} \int_{B(0,1)} | v_j(x) - v_j(x, t) | \, dx,$$

which is 0 by (10); and (11) is established.

Next, we observe from (11) and (3) that the Lebesgue measurable set Q defined by

$$(12)\qquad Q = \left\{ r: \int_{S(0,r)} v(x) \cdot n(x) dS(x) = g(r) \quad \text{and} \quad r \text{ in } P \right\}$$

is such that

$$(13)\qquad \mu(Q) = 1.$$

In order to complete the proof of the theorem, we need the following fact which we shall establish next:

$$(14)\qquad \int_\rho^{2\rho} | g(r) | \, r^{-k} dr = o(\rho) \qquad \text{as } \rho \to 0.$$

To establish (14), we observe from (4), (5), (8), and (9) that $g(r)r^{-k}$ is a positive constant multiple of $r\int_0^\infty A_u(0) J_{(k+2)/2}(ur)(ur)^{-k/2} du$. By assumption, $A_u(0) = o(1)$ as $u \to \infty$. Consequently, for $0 < 2\rho < 1$,

$$\begin{aligned}\int_\rho^{2\rho} g(r)r^{-k}dr &= \int_0^{1/\rho} o(1)\left\{\int_\rho^{2\rho} r\,|J_{(k+2)/2}(ur)|\,(ur)^{-k/2}dr\right\}du \\ &\quad + \int_{1/\rho}^{\infty} o(1)\left\{\int_\rho^{2\rho} r\,|J_{(k+2)/2}(ur)|\,(ur)^{-k/2}dr\right\}du.\end{aligned} \tag{15}$$

Calling the first expression on the right side of the equality in (15) I'_ρ and the second I''_ρ and using the familiar facts that there are constants K' and K'' such that $|J_{(k+2)/2}(t)| \leqq K' t^{(k+2)/2}$ for $0<t\leqq 2$ and $|J_{(k+2)/2}(t)| \leqq K'' t^{-1/2}$ for $1\leqq t<\infty$, we see in turn that as $\rho\to 0$,

$$I'_\rho \leqq \int_0^{1/\rho} o(1)\left[\int_\rho^{2\rho} r dr\right] du = o(\rho)$$

and

$$I''_\rho \leqq \int_{1/\rho}^{\infty} o(1)\left[\int_\rho^{2\rho} r^{-(k-1)/2}dr\right] u^{-(k+1)/2}du = o(\rho);$$

(14) is consequently established.

We now proceed along the lines given in [2, p. 324]. From (14) it follows that

$$\int_{2^{-(n+1)}}^{2^{-n}} |g(r)|\,|B(0,r)|^{-1}dr = \epsilon_n 2^{-n} \qquad \text{for } n = 1, 2, \cdots \tag{15'}$$

where $\lim_{n\to\infty} \epsilon_n = 0$.

For $n=1, 2, \cdots$, we define the set E_n to be

$$E_n = \{r\colon |g(r)|\,|B(0,r)|^{-1} \geqq \epsilon_n^{1/2}, r \text{ in } Q, \text{ and } 2^{-(n+1)} \leqq r \leqq 2^{-n}\}. \tag{16}$$

Then it follows from (15′) that

$$\mu(E_n) \leqq \epsilon_n^{1/2} 2^{-n}. \tag{17}$$

We set $E = \bigcup_{n=1}^\infty E_n$ and observe from (17) that $\mu[E\cap(0, 2^{-j})]2^j\to 0$ as $j\to\infty$. But then (13) implies that

(18) $Q - E$ has 0 as a point of density from the right.

Recalling the definition of P, we obtain from (12) that

$$\int_{S(0,r)} |v_j(x)n_j(x)|\,dS(x) < \infty \tag{19}$$

for r in $Q - E$ and $j = 1, \cdots, k$.

Next, using (12) in conjunction with (16), we obtain that

$$|B(0, r)|^{-1} \int_{S(0,r)} v(x) \cdot n(x) dS \to 0 \tag{20}$$

as $r \to 0$ through the points of $Q-E$.

But (18), (19), and (20) imply that (i) and (ii) hold with Q replaced by $Q-E$ and $\alpha=0$. We conclude that ap div $v(0)=0$, and the proof to the theorem is complete.

In closing we point out that for $k \geqq 4$ spherical convergence in the above theorem can be replaced by higher orders of Bochner-Riesz summability [**1**, p. 49].

Bibliography

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University of California, Riverside