THE APPROXIMATE DIVERGENCE OPERATOR

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1. We shall operate in Euclidean $k$-space, $k \geq 2$, and shall denote by $B(x, r)$ the $k$-ball with center $x$ and radius $r$. Similarly, by $S(x, r)$ we shall denote the $(k-1)$-sphere with center $x$ and radius $r$.

If $v(x) = [v_1(x), \ldots, v_k(x)]$ is a Lebesgue measurable vector field defined almost everywhere in $B(x_0, r_0)$ (that is each component is defined almost everywhere in $B(x_0, r_0)$ and is Lebesgue measurable), we shall say that the approximate divergence of $v$ exists at the point $x_0$ and equals $\alpha$ providing the following two facts hold:

(i) There exists a 1-dimensional Lebesgue measurable set $Q$ situated on the positive real axis which has 0 as a point of density from the right such that for $r$ in $(0, r_0) \cap Q$

$$\int_{S(x_0, r)} |v_j(x)n_j(x)| \, dS(x) < \infty, \quad j = 1, \ldots, k,$$

where $n(x) = [n_1(x), \ldots, n_k(x)]$ is the outward pointing unit normal to $S(x_0, r)$ at the point $x$ and $dS(x)$ is the natural $(k-1)$-dimensional volume element on $S(x_0, r)$.

(ii) As $r \to 0$ through the points of $Q$,

$$\left| B(x_0, r) \right|^{-1} \int_{S(x_0, r)} v(x) \cdot n(x) \, dS(x) \to \alpha$$

where $\left| B(x_0, r) \right|$ stands for the $k$-volume of $B(x_0, r)$.

If (i) and (ii) hold, we shall henceforth write $\text{ap \ div} \ v(x_0) = \alpha$.

It is clear that if $v(x)$ is in class $C^1$ in $B(x_0, r_0)$, then $\text{ap \ div} \ v(x_0)$ exists and equals the usual divergence of $v$ evaluated at $x_0$.

With an integral lattice point designated by $m$ where $m = (m_1, \ldots, m_k)$ and using the notation $(m, x) = m_1x_1 + \cdots + m_kx_k$ and $|m| = (m, m)^{1/2}$, we shall say the $k$-dimensional trigonometric series $\sum a_m e^{i(m, x)}$ is in class $D$ if for each $j$ there is a function $v_j(x)$ in $L^1(T_k)$ which has

$$\sum' - ia_m m_j |m|^{-2} e^{i(m, x)}$$

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as its Fourier series. If this is the case, we shall write

\[ v_j(x) \sim \sum' -i a_m z_j m^{-\beta} e^{i(m,x)}, \quad j = 1, \ldots, k, \]

and set

\[ v(x) = [v_1(x), \ldots, v_k(x)]. \]

\( T_k \), as usual, designates the \( k \)-dimensional torus, and \( \sum' \) the fact that the 0-lattice point is not to be considered. For multiple trigonometric series, we shall use the notation of [1].

We shall say that \( \sum a_m e^{i(m,x)} \) is spherically convergent at the point \( x_0 \) to \( a \) if

\[ \lim_{\|m\| \to \infty} \sum_{|m| \leq \alpha} a_m e^{i(m,x_0)} = a. \]

In this paper, we shall establish the following theorem:

**Theorem.** Let \( \sum a_m e^{i(m,x)} \) be a \( k \)-dimensional trigonometric series in class \( D \) which is spherically convergent at the point \( x_0 \) to \( a \) (of finite modulus). Then

\[ \text{ap div } v(x_0) = a - a_0, \]

where \( v(x) \) is the vector field defined by (2).

A close look at the definitions given will show that the above theorem is a generalization (and an improvement) of the theorem given in [2, p. 324].

2. We now proceed with the proof of the theorem. Without loss of generality, we assume that \( x_0 \) is the origin and that both \( a_0 = 0 \) and \( a = 0 \).

For each \( j \), we are given \( v_j(x) \) in \( L^1(T_k) \) satisfying (1). We define the 1-dimensional set \( P \) to be

\[ P = \{ r: \int_{S(0,r)} |v_j(x)| dS(x) < \infty \text{ for } j = 1, \ldots, k \text{ and } 0 < r < 1 \}. \]

Since for each \( j \), \( v_j(x) \) is also in \( L^1 \) on \( B(0,1) \), it follows from Fubini's theorem that \( P \) is a Lebesgue measurable set and furthermore that

\[ \mu(P) = 1 \quad \text{where } \mu \text{ is 1-dimensional Lebesgue measure}. \]

Next we set \( A_u(x) = \sum_{|m| \leq u} a_m e^{i(m,x)} \). Observing that \( d [J_p(t)t^{-p}] / dt = -J_{p+1}(t)t^{-p} \), where \( J_p(t) \) is the Bessel function of the first kind and order \( p \), and recalling that \( a_0 = 0 \), we obtain that for \( r > 0 \)
\begin{align*}
\sum_{|m| \leq u} a_m J_{k/2}(\frac{|m|}{u}) (\frac{|m|}{r})^{k/2} &= r \int_0^u A_1(0) J_{(k+2)/2} (tr) (tr)^{-k/2} dt + A_u(0) J_{k/2}(ur)(ur)^{-k/2}.
\end{align*}

Since both \( J_{k/2}(u) \) and \( J_{(k+2)/2}(u) \) are \( O(u^{-1/2}) \) as \( u \to \infty \) and since by assumption \( A_u(0) = o(1) \) as \( u \to \infty \), we conclude that

\begin{align}
\lim_{u \to \infty} \sum_{|m| \leq u} a_m J_{k/2}(\frac{|m|}{u}) (\frac{|m|}{r})^{k/2} e^{-|m| t} = r \int_0^\infty A_u(0) J_{(k+2)/2}(ur)(ur)^{-k/2} du.
\end{align}

The integral on the right side of the equality in (4) is absolutely convergent for every \( r > 0 \); consequently, the limit on the left side of the equality is finite for every \( r > 0 \). We obtain, therefore, that

\begin{align}
\lim_{t \to 0} \sum_{|m| \leq u} a_m J_{k/2}(\frac{|m|}{r}) (\frac{|m|}{r})^{k/2} e^{-|m| t} = \lim_{u \to \infty} \sum_{|m| \leq u} a_m J_{k/2}(\frac{|m|}{u}) (\frac{|m|}{r})^{k/2} e^{-|m| t} \quad \text{for } r > 0.
\end{align}

Next, we set for \( t > 0 \)

\begin{align}
v(x, t) &= \sum_m' - i a_m m_j \frac{|m|}{r} e^{i(m, x)} e^{-|m| t}.
\end{align}

and

\begin{align}
v(x, t) &= [v_1(x, t), \ldots, v_k(x, t)].
\end{align}

From a familiar computation involving Bessel functions, we obtain

\begin{align}
|B(0, r)|^{-1} \int_{S(0, r)} v(x, t) \cdot n(x) dS(x) = \gamma_k \sum_{m} a_m J_{k/2}(\frac{|m|}{r}) (\frac{|m|}{r})^{k/2} e^{-|m| t},
\end{align}

where \( \gamma_k \) is a positive constant depending on \( k \).

From (4), (5), and (8), it follows that, for \( 0 < r < 1 \), \( g(r) \) exists and is finite where \( g(r) \) is defined by the following limit

\begin{align}
g(r) = \lim_{t \to 0} \int_{S(0, r)} v(x, t) \cdot n(x) dS(x).
\end{align}
We note that for \( t > 0 \), \( v(x, t) \) is a continuous vector-valued function in \( x \); so \( g(r) \) is a Borel measurable function of \( r \) on the interval \((0, 1)\). We recall that \( v_j(x) \) is a function in \( L^1(T_k) \) satisfying (1). It follows therefore from (6) and well-known facts concerning multiple Fourier series, e.g. [1, (17), p. 56], that

\[
(10) \quad \int_{T_k} \left| v_j(x) - v_j(x, t) \right| \, dx \to 0 \quad \text{as } t \to 0 \quad \text{for } j = 1, \ldots, k.
\]

Using these facts, we next show

\[
(11) \quad \int_0^1 \left| \int_{S(0, r)} v(x) \cdot n(x) \, dS(x) - g(r) \right| \, dr = 0.
\]

We do this by observing that, by Fatou's lemma and (9), the expression on the left side of (11) is majorized by

\[
\liminf_{t \to 0} \int_{S(0, r)} \left| v(x) - v(x, t) \right| \cdot n(x) \, dS(x) \, dr.
\]

But this expression in turn is majorized by

\[
\liminf_{t \to 0} \sum_{j=1}^k \int_{B(0,1)} \left| v_j(x) - v_j(x, t) \right| \, dx,
\]

which is 0 by (10); and (11) is established.

Next, we observe from (11) and (3) that the Lebesgue measurable set \( Q \) defined by

\[
(12) \quad Q = \left\{ r : \int_{S(0, r)} v(x) \cdot n(x) \, dS(x) = g(r) \quad \text{and} \quad r \in P \right\}
\]

is such that

\[
(13) \quad \mu(Q) = 1.
\]

In order to complete the proof of the theorem, we need the following fact which we shall establish next:

\[
(14) \quad \int_0^{2\rho} |g(r)| \, r^{-k} \, dr = o(\rho) \quad \text{as } \rho \to 0.
\]

To establish (14), we observe from (4), (5), (8), and (9) that \( g(r) r^{-k} \) is a positive constant multiple of \( r \int_0^\infty A_u(0) J_{(k+2)\pi}(ur) (ur)^{-k/2} \, du \). By assumption, \( A_u(0) = o(1) \) as \( u \to \infty \). Consequently, for \( 0 < 2\rho < 1 \),
\[ \int_0^{2\rho} g(r) r^{-k} dr = \int_0^{1/\rho} o(1) \left\{ \int_0^{2\rho} r J_{(k+2)/2}(ur) |(ur)^{-k/2} dr \right\} du + \int_0^{1/\rho} o(1) \left\{ \int_0^{2\rho} r J_{(k+2)/2}(ur) |(ur)^{-k/2} dr \right\} du. \] 

(15)

Calling the first expression on the right side of the equality in (15) \( I' \) and the second \( I'' \) and using the familiar facts that there are constants \( K' \) and \( K'' \) such that \( |J_{(k+2)/2}(t)| \leq K' t^{(k+2)/2} \) for \( 0 < t \leq 2 \) and \( |J_{(k+2)/2}(t)| \leq K'' t^{-1/2} \) for \( 1 \leq t < \infty \), we see in turn that as \( \rho \to 0 \),

\[ I'_p \leq \int_0^{1/\rho} o(1) \left[ \int_0^{2\rho} r dr \right] du = o(\rho) \]

and

\[ I''_p \leq \int_0^{2\rho} o(1) \left[ \int_0^{2\rho} r^{-(k-1)/2} dr \right] u^{-(k+1)/2} du = o(\rho); \]

(14) is consequently established.

We now proceed along the lines given in [2, p. 324]. From (14) it follows that

\[ \int_{2^{-n}}^{2^{-n-1}} g(r) |B(0, r)|^{-1} dr = \epsilon_n 2^{-n} \quad \text{for } n = 1, 2, \ldots \]

where \( \lim_{n \to \infty} \epsilon_n = 0 \).

For \( n = 1, 2, \ldots \), we define the set \( E_n \) to be

\[ E_n = \{ r : |g(r)| |B(0, r)|^{-1} \geq \epsilon_n^{1/2}, r \in Q, \text{ and } 2^{-(n+1)} \leq r \leq 2^{-n} \}. \]

(16)

Then it follows from (15') that

\[ \mu(E_n) \leq \epsilon_n^{1/2} 2^{-n}. \]

We set \( E = \bigcup_{n=1}^\infty E_n \) and observe from (17) that \( \mu[E \cap (0, 2^{-j})] \to 0 \) as \( j \to \infty \). But then (13) implies that

\[ Q - E \text{ has 0 as a point of density from the right.} \]

Recalling the definition of \( P \), we obtain from (12) that

\[ \int_{S(0, r)} |v_j(x) n_j(x)| dS(x) < \infty \]

for \( r \in Q - E \) and \( j = 1, \ldots, k \).
Next, using (12) in conjunction with (16), we obtain that

\begin{equation}
\left| B(0, r) \right|^{-1} \int_{S(0, r)} v(x) \cdot n(x) dS \to 0
\end{equation}

as \( r \to 0 \) through the points of \( Q - E \).

But (18), (19), and (20) imply that (i) and (ii) hold with \( Q \) replaced by \( Q - E \) and \( \alpha = 0 \). We conclude that \( \text{ap div} \, v(0) = 0 \), and the proof to the theorem is complete.

In closing we point out that for \( k \geq 4 \) spherical convergence in the above theorem can be replaced by higher orders of Bochner-Riesz summability [1, p. 49].

**Bibliography**


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