STRUCTURE OF HEREDITARILY INFINITE DIMENSIONAL SPACES[1]

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1. **Introduction.** The first example of an infinite-dimensional compact metric space with no positive-dimensional compact subsets was constructed in 1965 by D. W. Henderson [5], [6]. Henderson's construction answered a question asked by L. A. Tumarkin in 1926 (see [8]) and first posed in print by M. S. Mazurkiewicz in 1933 [7], and provided a counterexample to van Heemert's claim that each infinite-dimensional compact metric space contains 1- and 2-dimensional compact subsets [4]. Infinite-dimensional compact metric spaces with no positive-dimensional compact subsets have come to be called hereditarily infinite-dimensional, or simply HID.

Since Henderson's discovery of the existence of HID spaces, several variations of his construction have been given, of which perhaps the most intuitively appealing is the one given by R. H. Bing in [3]. However, none of the alternative constructions give a great deal of insight into the structure of an HID space.

In this paper, we study the structure of HID spaces and attempt to provide some additional insight into their nature.

From a result of Tumarkin [9], it follows that each HID space contains an infinite-dimensional (and hence HID) Cantor manifold. An infinite-dimensional Cantor manifold is an infinite-dimensional compact metric space which is not separated by any finite-dimensional subspace. The principal result of this paper is that every HID space can be expressed as the union of a collection of maximal HID Cantor manifolds together with the set of points which lie in no HID Cantor manifold in X.

In this paper, all spaces will be compact metric spaces.

A compact metric space will be referred to as a *compactum*. By the dimension of a space we will mean the Menger-Urysohn, or small inductive, dimension or any equivalent definition of dimension in this setting.

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We will be dealing with the Hilbert cube I^ω, which we will regard as being the countable infinite product of intervals

$$I^\omega = I_1 \times I_2 \times I_3 \times \cdots$$

where $I_j = [-1/2^j, 1/2^j]$. The metric on I^ω is euclidean.

We recall the following definitions: A continuum is a compact connected set. A continuum is *indecomposable* if it is not the sum of two proper subcontinua. A continuum is *hereditarily indecomposable* if each of its subcontinua is indecomposable.

2. **Hereditarily indecomposable HID continua.** It is clear that not every HID continuum is indecomposable since the one point union of two HID continua is HID, but clearly not indecomposable. However, we will show that hereditarily indecomposable HID continua exist; in fact, every HID continuum contains uncountably many mutually exclusive hereditarily indecomposable HID subcontinua. Before proving the theorem, we prove the following lemma.

LEMMA 1. *Let X be an HID continuum, and let $p \in X$. Then either the dimension of X at p is* 1, *or there exists an $\epsilon > 0$ such that for each neighborhood U of p with* diam $U < \epsilon$, Bd U *is infinite dimensional.*

PROOF. Since X is connected, X has dimension greater than zero at each point. For every $p \in X$ and every neighborhood U of p such that Bd $U \neq \emptyset$, we must have dim Bd $U = 0$ or ∞, since in the contrary case Bd U would be a positive-dimensional compact subset of X, contrary to hypothesis. If the dimension of X at p is not 1, then the existence of the ϵ promised in the statement of the lemma is clear. The lemma is proved.

We are now in a position to prove Theorem 1. It should be noted that if we only wanted to show that each HID space contains a hereditarily indecomposable HID subcontinuum, we could do so without Lemma 1.

THEOREM 1. *Each HID continuum contains uncountably many mutually exclusive hereditarily indecomposable HID subcontinua.*

PROOF. It is a consequence of a theorem of Bing (Theorem 3 of [2]) that each point of I^ω lies in arbitrarily small neighborhoods whose boundaries are hereditarily indecomposable.

Let X be an HID continuum in I^ω and let $p \in X$ be chosen so that X has dimension ∞ at p. Let ϵ be a positive number (guaranteed by Lemma 1) such that Bd $S_\alpha(p)$ is infinite dimensional for $0 < \alpha < \epsilon$. ($S_\alpha(p)$ denotes the set of all points of X whose distance from p is less than α.) Let X_α be a component of Bd $S_\alpha(p)$ which is infinite dimensional (and therefore hereditarily infinite dimensional) and let $p_\alpha \in X_\alpha$ be a point at which X_α has dimension ∞. Let ϵ_α be a positive number such that if U_α is a neighborhood of p_α in X_α which has diameter $<\epsilon_\alpha$, then dim Bd $U_\alpha = \infty$. Let V_α be a neighborhood of p_α in I^ω whose boundary is hereditarily indecomposable, and whose diameter is less than ϵ_α. Finally, let Y_α be an infinite-dimensional component of Bd $V_\alpha \cap X_\alpha$. Then Y_α is a hereditarily indecomposable HID continuum for each α with $0<\alpha<\epsilon$, and the theorem is proved.

3. **HID Cantor manifolds.** Lemma 1 indicates that some HID spaces might be able to be separated by zero-dimensional sets. It would seem as though there should be an HID space which cannot be so separated. This is indeed true; it follows from a result of Tumarkin [9] that each HID space contains an infinite-dimensional (hence HID) Cantor manifold. We therefore have the following:

COROLLARY TO THEOREM 1. *Each HID space contains uncountably many mutually exclusive, hereditarily indecomposable HID Cantor manifolds.*

Since hereditarily indecomposable HID Cantor manifolds occur in abundance, we should perhaps point out that not every hereditarily indecomposable HID space is a Cantor manifold. We have the following proposition.

PROPOSITION 1. *There exist hereditarily indecomposable HID compacta which can be separated by* 0*-dimensional subsets.*

PROOF. Let X be a hereditarily indecomposable HID Cantor manifold, and let S be a separator of X. Let G be the monotone upper semicontinuous decomposition of X whose only nondegenerate elements are the components of S. Let $\pi: X \rightarrow X/G$ be the natural projection map. It is easily checked that $\pi(X)$ is hereditarily indecomposable since π is a monotone map, and X/G is separated by the 0-dimensional set $\pi(S)$. The proposition is proved.

We should also point out that HID Cantor manifolds need not be hereditarily indecomposable. Indeed, if X is an HID Cantor manifold and Y is a nondegenerate proper subcontinuum of X, then the space obtained by pasting two copies of X together along Y in the obvious manner is still an HID Cantor manifold, as can be easily checked, but it is clearly the sum of two proper subcontinua.

4. **Uncountably many topologically different HID continua.** We will construct an uncountable collection of HID continua, no

two of which are homeomorphic. This construction is based on the following two lemmas, the proofs of which are omitted. Lemma 2 is proved by an easy application of the Baire Category Theorem; the proof of Lemma 3 follows readily from a result of R. D. Anderson (Theorem 7.1 of [1]).

LEMMA 2. *Any compact space which is a countable union of HID compacta is an HID compactum.*

LEMMA 3. *Let X be an HID space, and let S be any finite subset of I^ω. Then X can be embedded in I^ω so that $S \subset X$, and in fact, the embedding may be chosen so that all points of S belong to the same component of X.*

We can now prove the following theorem.

THEOREM 2. *There are uncountably many topologically different HID continua.*

PROOF. Let X be a hereditarily indecomposable HID Cantor manifold in I^ω which contains the points $(-1/2, 0, 0, \cdots)$ and $(1/2, 0, 0, \cdots)$. We may further suppose that no other point of X has its first coordinate equal to $-1/2$ or $1/2$.

If $J = [a, b]$ is a subinterval of I^1, we define a homeomorphism h_J of I^ω into itself by defining

$$h_J(x_1, x_2, x_3, \cdots) = ((b - a)x_1 + (b + a)/2, (b - a)x_2, (b - a)x_3, \cdots).$$

Intuitively, h_J shrinks I^ω linearly and sends I^1 onto J. Let

$$J_i = [1/(i + 2), 1/(i + 1)], \qquad i = 1, 2, 3, \cdots.$$

We define an HID continuum Y as follows:

$$Y = (0, 0, 0, \cdots) \cup \bigcup_{i=1}^{\infty} h_{J_i}(X).$$

From Lemma 2, it is evident that Y is an HID continuum. (It should be noted that Y is 1-dimensional at the point $(0, 0, 0, \cdots)$ but is not 1-dimensional at any other point.) It is also clear that every homeomorphism of Y onto itself must take each of the points $(0, 0, 0, \cdots)$ and $(1/(i+1), 0, 0, \cdots)$, $i = 1, 2, 3, \cdots$, to itself.

Let $t \in [0, 1]$. Then t has a binary representation of the form $.b_1b_2b_3 \cdots$ where $b_i = 0$ or 1, $i = 1, 2, 3, \cdots$. Let Y_t be the HID space formed as follows: We adjoin a homeomorphic copy of X of diameter $1/i$ to Y at the point $1/(i+1)$ if and only if $b_i = 1$, $i = 1, 2, 3, \cdots$.

Then Y_t is homeomorphic to Y_s only if $t=s$, and there are uncountably many of the Y_t's. This completes the proof of the theorem.

5. **A structure theorem for HID continua.** We are now in a position to prove the principal result of this paper, which was mentioned in §1.

THEOREM 3. *Let X be an HID continuum, and let A be the set of points of X which lie in no HID Cantor manifold in X. Then*

$$X = A \cup \bigcup_{p \in X-A} M(p)$$

where each $M(p)$ is a maximal HID Cantor manifold containing p. If p, q are two points of $X-A$, then either $M(p)=M(q)$ or dim $(M(p)\cap M(q)) \leqq 0$.

PROOF. For each $p\in X$, let $\mathcal{C}_p$ be the family of all Cantor manifolds containing p, partially ordered by inclusion. $\mathcal{C}_p$ is not empty since $\{p\}$ is a Cantor manifold containing p. We observe that if $C\in\mathcal{C}_p$ is nondegenerate it is HID.

Let $\{C_\alpha\}$ be a totally ordered subcollection of $\mathcal{C}_p$. If $\bigcup_\alpha C_\alpha=\{p\}$, then $\bigcup_\alpha C_\alpha$ is a Cantor manifold of dimension 0. On the other hand, if x, y are two points of $\bigcup_\alpha C_\alpha$, then for some α_0, we have x, $y\in C_{\alpha_0}$. Since no finite-dimensional subset of C_{α_0} separates x from y, it follows that no finite-dimensional subset of $\bigcup_\alpha C_\alpha$ separates x from y. Hence if $\bigcup_\alpha C_\alpha$ is nondegenerate, its closure is an HID Cantor manifold. Thus each totally ordered subcollection of $\mathcal{C}_p$ has an upper bound, and, by Zorn's lemma, $\mathcal{C}_p$ contains a maximal element. Let $M(p)$ be a maximal element of $\mathcal{C}_p$.

To prove that if $M(p)\cap M(q)\neq\emptyset$ then either $M(p)=M(q)$ or $M(p)\cap M(q)$ is 0-dimensional, we first observe that if $M(p)\subset M(q)$ then $M(p)=M(q)$ by maximality of $M(p)$. Now if $M(p)\cap M(q)\neq\emptyset$ and $M(p)\neq M(q)$, then $M(p)\cup M(q)$ cannot be a Cantor manifold, again by maximality. Thus there is a 0-dimensional subset Z of $M(p)\cup M(q)$ which separates $M(p)\cup M(q)$. Since Z separates neither $M(p)$ nor $M(q)$, we have $(M(p)\cup M(q))-Z=A\cup B$ (separated), with $M(p)-Z\subset A$ and $M(q)-Z\subset B$. Then $M(p)\subset A\cup Z$ and $M(q)\subset B\cup Z$; hence $M(p)\cap M(q)\subset Z$. But Z is, by assumption, 0-dimensional. This completes the proof of the theorem.

6. **Questions.** The following questions are of interest in the further study of the structure of HID spaces:

(a) Does there exist a homogeneous HID space?

(b) Does there exist a homologically trivial HID space? (If so, this would answer a question raised by Bing in [3].)

(c) Do there exist uncountably many topologically different HID Cantor manifolds? Do there exist uncountably many topologically different hereditarily indecomposable HID Cantor manifolds?

(d) If X is an HID continuum and Y is an HID Cantor manifold in X, and if X is decomposed as in Theorem 3, is it necessarily true that $Y \subset M(p)$ for some $p \in X$?

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