WEAK COMPACTNESS OF MEASURES[1]

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1. **Introduction.** This paper is concerned with a description of the weakly relatively compact subsets of the space of regular Borel measures on a compact Hausdorff space X. Several characterizations of such sets are known through the work of Pettis [**6**], Grothendieck [**4**], and Dieudonné [**2**]. We find a weak set of Boolean conditions on a family of open sets of X to insure that convergence of a sequence of measures on each member of the family implies weak convergence of the sequence. This result is then applied to the Boolean algebra of regular open sets of X to obtain a generalization to arbitrary compact Hausdorff spaces of a theorem of Grothendieck on Stonian spaces.

W. G. Bade has remarked that Grothendieck's theorem is equivalent to a well-known lemma of R. S. Phillips concerning the equivalence of weak convergence and weak-star convergence in l_∞^*. Thus our generalization provides a new proof of Phillips' Lemma.

2. **Preliminaries.** Let X be a compact Hausdorff space. Denote the Banach space of all real or complex-valued continuous functions on X by $C(X)$, and denote the Banach space of all regular Borel measures on X by $M(X)$. The dual space of $C(X)$ is $M(X)$, and if $\mu \in M(X)$ then $\|\mu\| = |\mu|(X)$, the total variation of μ on X. The topology for $M(X)$ of pointwise convergence on $C(X)$ is called the weak star topology and is denoted by $\sigma(M(X), C(X))$. The topology for $M(X)$ of pointwise convergence on $M(X)^*$, the dual of $M(X)$, is called the weak topology and is denoted by $\sigma(M(X), M(X)^*)$. From the Eberlein-Smulian Theorem we know that a subset K of $M(X)$ is weakly relatively compact iff every sequence in K has a weakly converging subsequence. Also useful is the fact that $M(X)$ is a weakly complete space. The classical necessary and sufficient condition for weak convergence of a bounded sequence $\{\mu_n\}$ in $M(X)$ is that $\lim_n \mu_n(E)$ exists for each Borel set $E \subseteq X$, [**3**, p. 308]. Two basic results in this connection are

THEOREM 1 (GROTHENDIECK [**4**, p. 147]). *A sequence $\{\mu_n\}$ in $M(X)$ is $\sigma(M(X), M(X)^*)$-convergent iff for every sequence $\{E_n\}$ of pairwise disjoint open sets of X $\lim_m \mu_n(E_m) = 0$ uniformly in n.*

Received by the editors September 12, 1967.

[1] This work is part of a Ph.D. thesis written under the direction of Professor W. G. Bade at Berkeley. It was supported by the National Science Foundation Grant GP-5138.

THEOREM 2 (DIEUDONNÉ-GROTHENDIECK). *A sequence* $\{\mu_n\}$ *in* $M(X)$ *is convergent for the* $\sigma(M(X), M(X)^*)$ *topology iff for each open set* $G\subseteq X$, $\lim_n \mu_n(G)$ *exists.*

REMARK. Theorem 2 was first proved by Dieudonné [2] for X metric and later by Grothendieck for X an arbitrary compact Hausdorff space.

3. Main results.

DEFINITION. Let $\mathfrak{B}$ be a family of Borel sets of X.

(a) We call $\mathfrak{B}$ a *weak converging class* for $M(X)$ provided every sequence $\{\mu_n\}$ in $M(X)$ which converges for each member E of $\mathfrak{B}$ (i.e. $\{\mu_n(E)\}$ is a convergent sequence) converges for the weak topology.

(b) We call $\mathfrak{B}$ a *bounding class* for $M(X)$ provided every sequence $\{\mu_n\}$ in $M(X)$ which is bounded on each member E of $\mathfrak{B}$ (i.e. $\sup_n |\mu_n(E)| < \infty$) is such that its sequence of norms $\{\|\mu_n\|\}$ is bounded.

Theorem 2 states that the family $\mathfrak{B}$ of open sets of X is a weak converging class. Dieudonné also proved that it is a bounding class. Our first theorem gives a set of sufficient conditions on a family of open sets that it be both a weak converging class and a bounding class. We will apply our theorem to show that the regular open sets form a weak converging and a bounding class.

THEOREM 3. *Let* $\mathfrak{B}$ *be a family of open sets of a compact Hausdorff space* X, *and let* $\mathfrak{B}$ *satisfy*

(1) $\mathfrak{B}$ *is a basis for the topology of* X;

(2) *If* E_1 *and* E_2 *are in* $\mathfrak{B}$, *then* $E_1\cap E_2$ *is in* $\mathfrak{B}$;

(3) *If* E_1 *and* E_2 *are in* $\mathfrak{B}$, *and* $\overline{E}_1\cap\overline{E}_2=\varnothing$, *then* $E_1\cup E_2$ *is in* $\mathfrak{B}$.

(4) *If* K *is compact, and* U *is open, and* $K\subseteq U$, *then there exists an* E *in* $\mathfrak{B}$ *such that* $K\subseteq E\subseteq\overline{E}\subseteq U$;

(5) *If* $\{E_n\}$ *and* $\{G_n\}$ *are sequences from* $\mathfrak{B}$ *such that* $E_1\subseteq E_2\subseteq \cdots \subseteq E_n \cdots \subseteq G_n\subseteq \cdots G_2\subseteq G_1$, *then there is some* E_0 *in* $\mathfrak{B}$ *such that* $E_n\subseteq E_0\subseteq G_n$ *for every* n (E_0 *is said to interpolate the sequences*); *then* $\mathfrak{B}$ *is a weak converging family for* $M(X)$.

First we need the following

LEMMA 1. *Let* $\{E_n\}$ *be a sequence from a family of open sets* $\mathfrak{B}$ *satisfying conditions* (1)–(5), *and suppose that* $\mathrm{Cl}(\bigcup_{i\neq n} E_i)\cap\overline{E}_n=\varnothing$ *for each* n. *If* ν *is any nonnegative regular Borel measure, then for every* $\delta>0$ *there is an infinite set* A *of positive integers and an* $E_A\in\mathfrak{B}$ *such that* $\bigcup_{i\in A} E_i\subseteq E_A$ *and* $\nu(E_A)<\delta$.

PROOF. Let $\delta>0$ be given. We begin by choosing for each n an open set $U_{2n}\supseteq\overline{E}_{2n}$ such that $U_{2n}\cap\mathrm{Cl}(\bigcup_{i=1}^{\infty} E_{2i+1})=\emptyset$. Property (4) allows us to pick for each n a set $F_{2n}\in\mathfrak{B}$ such that $(\overline{E}_{2n})'\supseteq F_{2n}\supseteq U'_{2n}$.

Note that

$$E_1\subseteq E_1\cup E_3\subseteq E_1\cup E_3\cup E_5\subseteq\cdots\subseteq$$
$$\cdots\subseteq F_2\cap F_4\cap F_6\subseteq F_2\cap F_4\subseteq F_2.$$

By (5) there is some $G_1\in\mathfrak{B}$ which interpolates the sequences; G_1 has the properties that $G_1\subseteq\bigcup_{i=1}^{\infty} E_{2i+1}$ and $G_1\cap\bigcup_{i=1}^{\infty} E_{2i}=\emptyset$.

If now $\nu(G_1)<\delta$ we are done. If $\nu(G_1')<\delta/2$ then, since G_1' is compact and ν is a regular measure, it follows from (4) that there is some $F\in\mathfrak{B}$ such that $F\supseteq G_1'\supseteq\bigcup_{i=1}^{\infty} E_{2i}$, and is such that $\nu(F\cap G_1)<\delta/2$. Hence

$$\begin{aligned}\nu(F) &= \nu(F\cap G_1')+\nu(F\cap G_1)\\ &= \nu(G_1')+\nu(F\cap G_1)<\delta/2+\delta/2=\delta\end{aligned}$$

and we would be done.

If neither $\nu(G_1)<\delta$ nor $\nu(G_1')<\delta/2$, then we may repeat the above process to find disjoint subsequences $\{E_{n_i}\}$ and $\{E_{m_i}\}$ of the sequence $\{E_{2n+1}\}$ and a G_2 in $\mathfrak{B}$ such that $G_2\supset\bigcup_{i=1}^{\infty} E_{n_i}$, $G_2\cap\bigcup_{i=1}^{\infty} E_{m_i}=\emptyset$, and $G_2\subseteq G_1$. If now $\nu(G_2)<\delta$ we would be done. If $\nu(G_1\cap G_2')<\delta/2$, then we may pick an $H\in\mathfrak{B}$ such that $H\supseteq G_2'$ and $\nu(H\cap G_2)<\delta/2$. Note that

$$G_1\cap H=(G_1\cap G_2')\cup(G_2\cap H);$$

hence

$$\nu(G_1\cap H)=\nu(G_1\cap G_2')+\nu(G_2\cap H)<\delta/2+\delta/2=\delta.$$

We would be done since $G_1\cap H\in\mathfrak{B}$ by (2), and $G_1\cap H\supseteq\bigcup_{i=1}^{\infty} E_{m_i}$.

If neither $\nu(G_2)<\delta$ nor $\nu(G_1\cap G_2')<\delta/2$, then the above process may be repeated to get a $G_3\in\mathfrak{B}$, $G_3\subseteq G_2$. If this process does not terminate we would be able to find a decreasing sequence $\{G_n\}$ in $\mathfrak{B}$ with the property that $\nu(G_1)\geqq\delta$, $\nu(G_1')\geqq\delta/2$; $\nu(G_2)\geqq\delta$, $\nu(G_1\cap G_2')\geqq\delta/2$; $\cdots$; $\nu(G_n)\geqq\delta$, $\nu(G_{n-1}\cap G_n')\geqq\delta/2$; $\cdots$. However, the members of the sequence of sets G_1', $G_1\cap G_2'$, $G_2\cap G_3'$, $\cdots$ are pairwise disjoint. This would imply that the total variation of ν is infinite, which is a contradiction.

PROOF OF THEOREM 3. Let $\{\mu_n\}$ be a sequence of regular Borel measures converging on each member of $\mathfrak{B}$. To show that $\{\mu_n\}$ is a Cauchy sequence for the weak topology, it would suffice to show that $\{\mu_n-\mu_{n+p_n}\}$ converges to 0 in the weak topology for each sequence

$\{p_n\}$. Since $\{\mu_n(E)\}$ is convergent for each $E\in\mathcal{B}$, it follows that $\{(\mu_n-\mu_{n+p_n})(E)\}$ converges to 0 for each $E\in\mathcal{B}$. Hence, without loss of generality we may assume that $\{\mu_n\}$ is a sequence of regular Borel measures converging to 0 on each $E\in\mathcal{B}$ and prove that $\{\mu_n\}$ is weakly convergent to zero.

Assume to the contrary that $\{\mu_n\}$ is not weakly convergent. Then by (4), regularity of the measures, and Theorem I.1 it follows that there is a sequence $\{E_n\}$ from $\mathcal{B}$ and a positive ϵ such that $\mathrm{Cl}(\bigcup_{i\neq n} E_i)\cap\overline{E}_n=\emptyset$ for each n, such that a subsequence of $\{\mu_n\}$, without loss of generality still called $\{\mu_n\}$, satisfies $|\mu_n(E_n)|>\epsilon>0$.

We now carry out an inductive process to obtain a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ such that $\{\mu_n\}$ does not converge to zero on some $E_0\in\mathcal{B}$ such that $E_0\supseteq\bigcup_{i=1}^{\infty} E_{n_i}$. This will contradict the hypothesis that $\{\mu_n\}$ converges to zero on every member of $\mathcal{B}$.

We apply the lemma to the measure $|\mu_1|$ to get an infinite set of positive integers A_1 and an $E_{A_1}\in\mathcal{B}$ such that $\overline{E}_{A_1}\cap\overline{E}_1=\emptyset$ and $E_{A_1}\supseteq\bigcup_{n\in A_1} E_n$ and $|\mu_1|(E_{A_1})<\epsilon/3$.

First set $n_0=1$ and pick $n_1\in A_1$ so large that $|\mu_n(E_1)|<\epsilon/3$ for all $n\geqq n_1$. Next apply the lemma again along with property (2) to extract an infinite set of positive integers A_2 from the set $\{A_1\cap\text{all integers}\geqq n_2\}$ and obtain an $E_{A_2}\subseteq E_{A_1}$, $E_{A_2}\in\mathcal{B}$, such that $E_{A_2}\supseteq\bigcup_{n\in A_2}E_n$, $\overline{E}_{A_2}\cap\overline{E}_{n_1}=\emptyset$, and $|\mu_{n_1}|(E_{A_2})<\epsilon/3$.

Now pick $n_2>n_1$, $n_2\in A_2$, so large that $|\mu_n(E_1)|+|\mu_n(E_{n_1})|<\epsilon/3$ for all $n\geqq n_2$. Continuing in this fashion we obtain a sequence of integers $\{n_0, n_1, n_2, \cdots\}$ and a decreasing sequence of sets in $\mathcal{B}$, $E_{A_1}\supseteq E_{A_2}\supseteq E_{A_3}\supseteq\cdots$ such that the following hold:

(a) $$|\mu_{n_i}|(E_{A_{i+1}}) < \epsilon/3 \qquad \text{for all } i,$$

(b) $$\sum_{i=0}^{j-1} |\mu_n(E_{n_i})| < \epsilon/3 \qquad \text{for all } n \geqq n_j.$$

Consider the sequence of sets:

$$E_1\subseteq E_1\cup E_{n_1}\subseteq E_1\cup E_{n_1}\cup E_{n_2}\subseteq \cdots$$
$$\cdots\subseteq E_{A_3}\cup E_{n_2}\cup E_{n_1}\cup E_1\subseteq E_{A_2}\cup E_{n_1}\cup E_1\subseteq E_{A_1}\cup E_1.$$

Each member of the sequence is in $\mathcal{B}$, and by property (5) we may choose an E_0 from $\mathcal{B}$ which interpolates the sequence.

It simply remains to note that:

$$|\mu_{n_j}(E_0)| \geqq |\mu_{n_j}(E_{n_j})| - \sum_{i=1}^{j-1} |\mu_{n_j}(E_{n_i})| - |\mu_{n_j}|(E_{A_{j+1}})$$
$$\geqq \epsilon - \epsilon/3 - \epsilon/3 = \epsilon/3$$

holds for every j. However, this contradicts our earlier assumption that the sequence $\{\mu_n(E)\}$ converges to 0 for each member E of $\mathcal{B}$. Q.E.D.

COROLLARY. *If $\mathcal{B}$ is a family of open sets of a compact Hausdorff space X satisfying conditions* (1)–(5) *then $\mathcal{B}$ is a bounding class for $M(X)$.*

PROOF. Suppose $\{\mu_i\}$ is a sequence in $M(X)$ satisfying $\sup_i |\mu_i(E)| < \infty$ for every $E \in \mathcal{B}$. If $\{\|\mu_i\|\}$ is not bounded then without loss of generality we may drop to a subsequence and assume $\lim_i \|\mu_i\| = \infty$. Now we may multiply each μ_i by an appropriate scalar (e.g. $\|\mu_i\|^{1/2}/\|\mu_i\|$) to insure that $\lim_i \mu_i(E) = 0$ for every $E \in \mathcal{B}$ while maintaining $\lim_i \|\mu_i\| = \infty$. The proof of the theorem shows that $\{\mu_i\}$ is $\sigma(M(X), M^*(X))$—convergent to zero. However, this is impossible in view of $\lim_i \|\mu_i\| = \infty$. Q.E.D.

DEFINITION. An open set U is called *regular* if $U = \text{int}(\overline{U})$.

The set of regular open sets of a topological space when ordered by set inclusion is a complete Boolean algebra. The supremum of a family $(U_\alpha)_{\alpha \in A}$ of regular open sets, denoted by $\bigvee_{\alpha \in A} U_\alpha$, is defined to be $\text{int}(\text{Cl}(\bigcup_{\alpha \in A} U)$; the infimum, denoted by $\bigwedge_{\alpha \in A} U_\alpha$, is defined to be $\text{int}(\text{Cl}(\bigcap_{\alpha \in A} U_\alpha))$. The intersection of two regular open sets is regular. However, the union of two regular open sets need not be regular, and this fact presents the essential difficulty, since a Borel measure need not be even finitely additive with respect to the Boolean operations. However, if the closures of two regular open sets are disjoint then their union is regular. A complete discussion of regular open sets may be found in Halmos [**5**, p. 13].

THEOREM 4. *If X is a compact Hausdorff space and $\mathcal{B}$ is the Boolean algebra of all the regular open sets of X, then $\mathcal{B}$ is both a weak converging class and a bounding class.*

PROOF. $\mathcal{B}$ obviously satisfies conditions (1)–(4) of Theorem 3. Also if $E_1 \subseteq E_2 \subseteq E_3 \cdots \subseteq G_3 \subseteq G_2 \subseteq G_1$ is such that each member of the sequence is a regular open set, then both $\bigvee_{i=1}^\infty E_i$ and $\bigwedge_{i=1}^\infty G_i$ interpolate the sequence. Thus $\mathcal{B}$ satisfies the conditions of Theorem 3, and we conclude that $\mathcal{B}$ is both a weak converging class and a bounding class. Q.E.D.

REMARK. W. G. Bade and P. C. Curtis had previously shown (unpublished) that the regular open sets are a bounding class.

DEFINITION. A compact Hausdorff space X is called *Stonian* if the closure of every open set is open.

LEMMA 2. *X is Stonian iff every regular open set is open and closed.*

PROOF. If X is Stonian and U is a regular open set, i.e., $U=\operatorname{int}(\overline{U})$, then $\operatorname{int}(\overline{U})=\overline{U}$ since $\overline{U}$ is an open set. Hence $U=\overline{U}$ and U is open and closed. Conversely, if every regular open set is open and closed, then if U is an open set, since $U\subseteq\operatorname{int}(\overline{U})$, it follows that $\overline{U}\subseteq\operatorname{int}(\overline{U})$ and hence that $\overline{U}=\operatorname{int}(\overline{U})$. Thus $\overline{U}$ is an open set. Q.E.D.

Our Theorem 4 is a generalization to arbitrary compact Hausdorff spaces of the following theorem of A. Grothendieck [**4**, p. 168].

THEOREM 5. *Let X be Stonian and $\{\mu_n\}$ a sequence in $M(X)$. Then $\{\mu_n(E)\}$ is a convergent sequence for every open closed E iff $\{\mu_n\}$ is convergent for the $\sigma(M(X), M(X)^*)$ topology.*

PROOF. By Lemma 2 the regular open sets of X are precisely the open closed sets. Hence Theorem 4 gives us the result. Q.E.D.

NOTATION AND DEFINITIONS. Let S be a discrete set. Then βS denotes the Stone-Čech compactification of S. It is well known that βS is a Stonian space. The space of all bounded real or complex-valued functions on S with the supremum norm will be denoted by $B(S)$. The space of finitely additive measures on the field Σ of all subsets of S will be denoted by $ba(S, \Sigma)$. If $\mu\in ba(S, \Sigma)$ then $\|\mu\|=|\mu|(S)$, the total variation of μ on S. The *atomic* part of μ is defined by $\nu(E)=\sum_{s\in E}\mu(s)$ where $E\in\Sigma$. We shall need the facts that $C(\beta S)$ is isometrically isomorphic to $B(S)$ and $M(\beta S)$ is isometrically isomorphic to $ba(S, \Sigma)$. For a complete discussion of these facts see Dunford and Schwartz [**3**, p. 311–313].

Grothendieck's proof of Theorem 5 was based on the following result due to Phillips [**7**].

THEOREM 6. *Let S be a discrete set and $\{\mu_n\}$ a sequence in $ba(S, \Sigma)$. If $\{\mu_n(E)\}$ converges to 0 for each $E\in\Sigma$, then $\{\|\nu_n\|\}$ converges to 0, where ν_n is the atomic part of μ_n.*

REMARK (BADE). Theorem 5 is equivalent to Theorem 6.

PROOF. Assume Theorem 5 and that $\{\mu_n\}$ is a sequence in $ba(S, \Sigma)$ such that $\lim_n \mu_n(E)=0$ for every $E\in\Sigma$. Let $\bar{\mu}_n$ be the correspondent of μ_n in $M(\beta S)$, and $k_{\bar{E}}$ be the correspondent of k_E in $C(\beta S)$ (k_E denotes the characteristic function of E). Then $\bar{\mu}_n(k_{\bar{E}})=\mu_n(k_E)$, and it follows that $\bar{\mu}_n$ converges to 0 for each open closed set in βS. By Theorem 5 $\{\bar{\mu}_n\}$ converges to 0 in the $\sigma(M(\beta S), M(\beta S)^*)$ topology. Thus $\{\mu_n\}$ converges to 0 for the $\sigma(ba(S, \Sigma), ba(S, \Sigma)^*)$ topology.

Let P denote the projection of norm 1 of $ba(S, \Sigma)$ onto $l_1(S)$ defined by $P: \mu \to \nu$ where ν is the atomic part of μ. P is norm continuous and hence is continuous for the weak topologies. Thus $\{P\mu_n\} = \{\nu_n\}$ converges to 0 for the $\sigma(l_1(S), l_\infty(S))$ topology. By a theorem of Banach [1, p. 137] $\{\|\nu_n\|\}$ converges to zero.

Assume Theorem 6 and that $\{\mu_n\}$ is a sequence of regular Borel measures on a Stonian space S which converges to 0 on each open closed subset of S. To show that $\{\mu_n\}$ is weakly convergent to 0, it suffices to show (by Theorem 1) that $\{\mu_n(E_n)\}$ converges to 0, where $\{E_n\}$ is an arbitrary sequence of pairwise disjoint open closed subsets of S.

Define for each n a set function ν_n on N, the set of positive integers:

$$\nu_n(A) = \mu_n\left(\bigvee_{i \in A} E_i\right) \quad \text{where} \quad A \subseteq N.$$

Note that ν_n is bounded and finitely additive, and hence an element of $ba(N, \Sigma)$. Since $\{\mu_n\}$ converges to 0 on each open closed subset of S, $\{\nu_n(A)\}$ converges to 0 for each $A \in \Sigma$. Theorem 6 allows the conclusion that $\lim_n \sum_{i=1}^{\infty} |\nu_n(i)| = 0$. In particular

$$\lim_n |\nu_n(n)| = \lim_n |\mu_n(E_n)| = 0. \qquad \text{Q.E.D.}$$

The proof of Theorem 4 thus provides a new proof of Theorem 6.

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