WEAK COMPACTNESS OF MEASURES

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1. Introduction. This paper is concerned with a description of the weakly relatively compact subsets of the space of regular Borel measures on a compact Hausdorff space $X$. Several characterizations of such sets are known through the work of Pettis [6], Grothendieck [4], and Dieudonné [2]. We find a weak set of Boolean conditions on a family of open sets of $X$ to insure that convergence of a sequence of measures on each member of the family implies weak convergence of the sequence. This result is then applied to the Boolean algebra of regular open sets of $X$ to obtain a generalization to arbitrary compact Hausdorff spaces of a theorem of Grothendieck on Stonian spaces.

W. G. Bade has remarked that Grothendieck's theorem is equivalent to a well-known lemma of R. S. Phillips concerning the equivalence of weak convergence and weak-star convergence in $l^*_i$. Thus our generalization provides a new proof of Phillips' Lemma.

2. Preliminaries. Let $X$ be a compact Hausdorff space. Denote the Banach space of all real or complex-valued continuous functions on $X$ by $C(X)$, and denote the Banach space of all regular Borel measures on $X$ by $M(X)$. The dual space of $C(X)$ is $M(X)$, and if $\mu \in M(X)$ then $||\mu|| = |\mu|(X)$, the total variation of $\mu$ on $X$. The topology for $M(X)$ of pointwise convergence on $C(X)$ is called the weak star topology and is denoted by $\sigma(M(X), C(X))$. The topology for $M(X)$ of pointwise convergence on $M(X)^*$, the dual of $M(X)$, is called the weak topology and is denoted by $\sigma(M(X), M(X)^*)$. From the Eberlein-Smulian Theorem we know that a subset $K$ of $M(X)$ is weakly relatively compact iff every sequence in $K$ has a weakly converging subsequence. Also useful is the fact that $M(X)$ is a weakly complete space. The classical necessary and sufficient condition for weak convergence of a bounded sequence $\{\mu_n\}$ in $M(X)$ is that $\lim_n \mu_n(E)$ exists for each Borel set $E \subseteq X$, [3, p. 308]. Two basic results in this connection are

Theorem 1 (Grothendieck [4, p. 147]). A sequence $\{\mu_n\}$ in $M(X)$ is $\sigma(M(X), M(X)^*)$-convergent iff for every sequence $\{E_n\}$ of pairwise disjoint open sets of $X$ $\lim_n \mu_n(E_n) = 0$ uniformly in $n$. 

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Theorem 2 (Dieudonné-Grothendieck). A sequence $\{\mu_n\}$ in $M(X)$ is convergent for the $\sigma(M(X), M(X)^*)$ topology iff for each open set $G \subseteq X$, $\lim_n \mu_n(G)$ exists.

Remark. Theorem 2 was first proved by Dieudonné [2] for $X$ metric and later by Grothendieck for $X$ an arbitrary compact Hausdorff space.

3. Main results.

Definition. Let $\mathcal{B}$ be a family of Borel sets of $X$.

(a) We call $\mathcal{B}$ a weak converging class for $M(X)$ provided every sequence $\{\mu_n\}$ in $M(X)$ which converges for each member $E$ of $\mathcal{B}$ (i.e. $\{\mu_n(E)\}$ is a convergent sequence) converges for the weak topology.

(b) We call $\mathcal{B}$ a bounding class for $M(X)$ provided every sequence $\{\mu_n\}$ in $M(X)$ which is bounded on each member $E$ of $\mathcal{B}$ (i.e. $\sup_n |\mu_n(E)| < \infty$) is such that its sequence of norms $\{||\mu_n||\}$ is bounded.

Theorem 2 states that the family $\mathcal{B}$ of open sets of $X$ is a weak converging class. Dieudonné also proved that it is a bounding class.

Our first theorem gives a set of sufficient conditions on a family of open sets that it be both a weak converging class and a bounding class. We will apply our theorem to show that the regular open sets form a weak converging and a bounding class.

Theorem 3. Let $\mathcal{B}$ be a family of open sets of a compact Hausdorff space $X$, and let $\mathcal{B}$ satisfy

1. $\mathcal{B}$ is a basis for the topology of $X$;
2. If $E_1$ and $E_2$ are in $\mathcal{B}$, then $E_1 \cap E_2$ is in $\mathcal{B}$;
3. If $E_1$ and $E_2$ are in $\mathcal{B}$, and $E_1 \cap E_2 = \emptyset$, then $E_1 \cup E_2$ is in $\mathcal{B}$.
4. If $K$ is compact, and $U$ is open, and $K \subseteq U$, then there exists an $E$ in $\mathcal{B}$ such that $K \subseteq E \subseteq \overline{E} \subseteq U$;
5. If $\{E_n\}$ and $\{G_n\}$ are sequences from $\mathcal{B}$ such that $E_1 \subseteq E_2 \subseteq \ldots \subseteq E_n \ldots \subseteq G_n \subseteq \ldots \subseteq G_2 \subseteq G_1$, then there is some $E_0$ in $\mathcal{B}$ such that $E_n \subseteq E_0 \subseteq G_n$ for every $n$ ($E_0$ is said to interpolate the sequences); then $\mathcal{B}$ is a weak converging family for $M(X)$.

First we need the following

Lemma 1. Let $\{E_n\}$ be a sequence from a family of open sets $\mathcal{B}$ satisfying conditions (1)-(5), and suppose that $\text{Cl}(\bigcup_{i \in A} E_i) \cap \overline{E_n} = \emptyset$ for each $n$. If $\nu$ is any nonnegative regular Borel measure, then for every $\delta > 0$ there is an infinite set $A$ of positive integers and an $E_A \in \mathcal{B}$ such that $\bigcup_{i \in A} E_i \subseteq E_A$ and $\nu(E_A) < \delta$. 

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Proof. Let $\delta > 0$ be given. We begin by choosing for each $n$ an open set $U_{2n} \supseteq \overline{E}_{2n}$ such that $U_{2n} \cap \text{Cl} \left( \bigcup_{n=1}^{\infty} E_{2n+1} \right) = \emptyset$. Property (4) allows us to pick for each $n$ a set $F_{2n} \subseteq \mathfrak{B}$ such that $(\overline{E}_{2n})' \supseteq F_{2n} \supseteq U'_{2n}$.

Note that

$$E_1 \subseteq E_1 \cup E_3 \subseteq E_1 \cup E_3 \cup E_5 \subseteq \cdots \subseteq \cdots \subseteq F_2 \cap F_4 \cap F_6 \subseteq F_2 \cap F_4 \subseteq F_2.$$ 

By (5) there is some $G_1 \subseteq \mathfrak{B}$ which interpolates the sequences; $G_1$ has the properties that $G_1 \subseteq \bigcup_{i=1}^{\infty} E_{2i+1}$ and $G_1 \cap \bigcup_{i=1}^{\infty} E_{2i} = \emptyset$.

If now $\nu(G_1) < \delta$ we are done. If $\nu(G_1') < \delta/2$ then, since $G_1'$ is compact and $\nu$ is a regular measure, it follows from (4) that there is some $F \subseteq \mathfrak{B}$ such that $F \supseteq G_1' \supseteq \bigcup_{i=1}^{\infty} E_{2i}$, and is such that $\nu(F \cap G_1') < \delta/2$. Hence

$$\nu(F) = \nu(F \cap G_1') + \nu(F \cap G_1') < \delta/2 + \delta/2 = \delta.$$ 

and we would be done.

If neither $\nu(G_1') < \delta$ nor $\nu(G_1') < \delta/2$, then we may repeat the above process to find disjoint subsequences $\{E_{2n+i}\}$ and $\{E_{m_i}\}$ of the sequence $\{E_{2n+1}\}$ and a $G_2$ in $\mathfrak{B}$ such that $G_2 \supseteq \bigcup_{i=1}^{\infty} E_{m_i}$, $G_2 \cap \bigcup_{i=1}^{\infty} E_{m_i} = \emptyset$, and $G_2 \subseteq G_1$. If now $\nu(G_2) < \delta$ we would be done. If $\nu(G_1 \cap G_2') < \delta/2$, then we may pick an $H \subseteq \mathfrak{B}$ such that $H \supseteq G_2'$ and $\nu(H \cap G_2) < \delta/2$. Note that

$$G_1 \cap H = (G_1 \cap G_2') \cup (G_2 \cap H);$$ 

hence

$$\nu(G_1 \cap H) = \nu(G_1 \cap G_2') + \nu(G_2 \cap H) < \delta/2 + \delta/2 = \delta.$$ 

We would be done since $G_1 \cap H \subseteq \mathfrak{B}$ by (2), and $G_1 \cap H \supseteq \bigcup_{i=1}^{\infty} E_{m_i}$.

If neither $\nu(G_2) < \delta$ nor $\nu(G_1 \cap G_2') < \delta/2$, then the above process may be repeated to get a $G_3 \subseteq \mathfrak{B}, G_3 \subseteq G_2$. If this process does not terminate we would be able to find a decreasing sequence $\{G_n\}$ in $\mathfrak{B}$ with the property that $\nu(G_1) \geq \delta$, $\nu(G_1') \geq \delta/2$; $\nu(G_2) \geq \delta$, $\nu(G_1 \cap G_2') \geq \delta/2$; $\cdots$; $\nu(G_n) \geq \delta$, $\nu(G_{n-1} \cap G_n') \geq \delta/2$; $\cdots$. However, the members of the sequence of sets $G_1', G_1 \cap G_1', G_2 \cap G_1', \cdots$ are pairwise disjoint. This would imply that the total variation of $\nu$ is infinite, which is a contradiction.

Proof of Theorem 3. Let $\{\mu_n\}$ be a sequence of regular Borel measures converging on each member of $\mathfrak{B}$. To show that $\{\mu_n\}$ is a Cauchy sequence for the weak topology, it would suffice to show that $\{\mu_n - \mu_{n+p}\}$ converges to 0 in the weak topology for each sequence.
\{p_n\}$. Since \{\mu_n(E)\} is convergent for each $E \in \mathfrak{B}$, it follows that
\[(\mu_n - \mu_{n+p})\] converges to 0 for each $E \in \mathfrak{B}$. Hence, without loss of generality we may assume that \{\mu_n\} is a sequence of regular Borel measures converging to 0 on each $E \in \mathfrak{B}$ and prove that \{\mu_n\} is weakly convergent to zero.

Assume to the contrary that \{\mu_n\} is not weakly convergent. Then by (4), regularity of the measures, and Theorem I.1 it follows that there is a sequence \{E_n\} from \mathfrak{B} and a positive $\varepsilon$ such that $\text{Cl}(\bigcup_{i} E_i) \cap E_n = \emptyset$ for each $n$, such that a subsequence of \{\mu_n\}, without loss of generality still called \{\mu_n\}, satisfies $|\mu_n(E_n)| > \varepsilon > 0$.

We now carry out an inductive process to obtain a subsequence \{E_{n_i}\} of \{E_n\} such that \{\mu_n\} does not converge to zero on some $E \in \mathfrak{B}$ such that $E \supseteq \bigcup_{i} E_n$. This will contradict the hypothesis that \{\mu_n\} converges to zero on every member of \mathfrak{B}.

We apply the lemma to the measure $|\mu_i|$ to get an infinite set of positive integers $A_1$ and an $E_{A_1} \in \mathfrak{B}$ such that $\overline{E_{A_1}} \cap E_1 = \emptyset$ and $E_{A_1} \supseteq \bigcup_{i} E_i$ and $|\mu_i|(E_{A_1}) < \varepsilon/3$.

First set $n_0 = 1$ and pick $n_1 \in A_1$ so large that $|\mu_n(E_1)| < \varepsilon/3$ for all $n \geq n_1$. Next apply the lemma again along with property (2) to extract an infinite set of positive integers $A_2$ from the set \{all integers $\geq n_1$\} and obtain an $E_{A_2} \subseteq E_{A_1}$, $E_{A_2} \in \mathfrak{B}$, such that $E_{A_2} \supseteq \bigcup_{i} E_i$, $E_{A_2} \cap E_{A_1} = \emptyset$, and $|\mu_n|(E_{A_2}) < \varepsilon/3$.

Now pick $n_2 > n_1$, $n_2 \in A_2$, so large that $|\mu_n(E_1)| + |\mu_n(E_{n_1})| < \varepsilon/3$ for all $n \geq n_2$. Continuing in this fashion we obtain a sequence of integers \{n_0, n_1, n_2, \cdots\} and a decreasing sequence of sets in \mathfrak{B}, $E_{A_1} \supseteq E_{A_2} \supseteq \cdots$ such that the following hold:

(a) $|\mu_{n_i}|(E_{A_{i+1}}) < \varepsilon/3$ for all $i$,
(b) $\sum_{i=0}^{j-1} |\mu_{n_i}(E_{n_{i+1}})| < \varepsilon/3$ for all $n \geq n_j$.

Consider the sequence of sets:

$E_1 \subseteq E_1 \cup E_{n_1} \subseteq E_1 \cup E_{n_1} \cup E_{n_2} \subseteq \cdots$

$\cdots \subseteq E_A \cup E_{n_A} \cup E_{n_1} \cup E_1 \subseteq E_{A_2} \cup E_{n_1} \cup E_1 \subseteq E_{A_1} \cup E_1$.

Each member of the sequence is in \mathfrak{B}, and by property (5) we may choose an $E_0$ from \mathfrak{B} which interpolates the sequence.

It simply remains to note that:

\[|\mu_{n_j}(E_0)| \geq |\mu_{n_j}(E_{n_j})| - \sum_{i=1}^{j-1} |\mu_{n_j}(E_{n_{i+1}})| - |\mu_{n_j}|(E_{A_{j+1}}) \geq \varepsilon - \varepsilon/3 - \varepsilon/3 = \varepsilon/3\]
holds for every $j$. However, this contradicts our earlier assumption that the sequence $\{\mu_n(E)\}$ converges to 0 for each member $E$ of $\emptyset$. Q.E.D.

**Corollary.** If $\emptyset$ is a family of open sets of a compact Hausdorff space $X$ satisfying conditions (1)–(5) then $\emptyset$ is a bounding class for $M(X)$.

**Proof.** Suppose $\{\mu_i\}$ is a sequence in $M(X)$ satisfying $\sup E |\mu_i(E)| < \infty$ for every $E \in \emptyset$. If $\{\|\mu_i\|\}$ is not bounded then without loss of generality we may drop to a subsequence and assume $\lim_i \|\mu_i\| = \infty$. Now we may multiply each $\mu_i$ by an appropriate scalar (e.g. $\|\mu_i\|^{1/2}/\|\mu_i\|$) to insure that $\lim_i |\mu_i(E)| = 0$ for every $E \in \emptyset$ while maintaining $\lim_i \|\mu_i\| = \infty$. The proof of the theorem shows that $\{\mu_i\}$ is $\sigma(M(X), M^*(X))$—convergent to zero. However, this is impossible in view of $\lim_i \|\mu_i\| = \infty$. Q.E.D.

**Definition.** An open set $U$ is called regular if $U = \text{int}(U)$.

The set of regular open sets of a topological space when ordered by set inclusion is a complete Boolean algebra. The supremum of a family $(U_a)_{a \in A}$ of regular open sets, denoted by $\bigvee_{a \in A} U_a$, is defined to be $\text{int}(\overline{\bigcup_{a \in A} U_a})$; the infimum, denoted by $\bigwedge_{a \in A} U_a$, is defined to be $\text{int}(\overline{\bigcap_{a \in A} U_a})$. The intersection of two regular open sets is regular. However, the union of two regular open sets need not be regular, and this fact presents the essential difficulty, since a Borel measure need not be even finitely additive with respect to the Boolean operations. However, if the closures of two regular open sets are disjoint then their union is regular. A complete discussion of regular open sets may be found in Halmos [5, p. 13].

**Theorem 4.** If $X$ is a compact Hausdorff space and $\emptyset$ is the Boolean algebra of all the regular open sets of $X$, then $\emptyset$ is both a weak converging class and a bounding class.

**Proof.** $\emptyset$ obviously satisfies conditions (1)–(4) of Theorem 3. Also if $E_1 \subseteq E_2 \subseteq E_3 \cdots \subseteq G_3 \subseteq G_2 \subseteq G_1$ is such that each member of the sequence is a regular open set, then both $\bigvee_{i=1}^n E_i$ and $\bigwedge_{i=1}^n G_i$ interpolate the sequence. Thus $\emptyset$ satisfies the conditions of Theorem 3, and we conclude that $\emptyset$ is both a weak converging class and a bounding class. Q.E.D.

**Remark.** W. G. Bade and P. C. Curtis had previously shown (unpublished) that the regular open sets are a bounding class.

**Definition.** A compact Hausdorff space $X$ is called Stonian if the closure of every open set is open.
Lemma 2. X is Stonian iff every regular open set is open and closed.

Proof. If X is Stonian and $U$ is a regular open set, i.e., $U = \text{int}(\overline{U})$, then $\text{int}(\overline{U}) = U$ since $U$ is an open set. Hence $U = \overline{U}$ and $U$ is open and closed. Conversely, if every regular open set is open and closed, then if $U$ is an open set, since $U \subseteq \text{int}(\overline{U})$, it follows that $\overline{U} \subseteq \text{int}(\overline{U})$ and hence that $\overline{U} = \text{int}(\overline{U})$. Thus $\overline{U}$ is an open set. Q.E.D.

Our Theorem 4 is a generalization to arbitrary compact Hausdorff spaces of the following theorem of A. Grothendieck [4, p. 168].

Theorem 5. Let $X$ be Stonian and $\{\mu_n\}$ a sequence in $M(X)$. Then $\{\mu_n(E)\}$ is a convergent sequence for every open closed $E$ iff $\{\mu_n\}$ is convergent for the $\sigma(M(X), M(X)^*)$ topology.

Proof. By Lemma 2 the regular open sets of $X$ are precisely the open closed sets. Hence Theorem 4 gives us the result. Q.E.D.

Notation and Definitions. Let $S$ be a discrete set. Then $\beta S$ denotes the Stone-Čech compactification of $S$. It is well known that $\beta S$ is a Stonian space. The space of all bounded real or complex-valued functions on $S$ with the supremum norm will be denoted by $B(S)$. The space of finitely additive measures on the field $\Sigma$ of all subsets of $S$ will be denoted by $ba(S, \Sigma)$. If $\mu \in ba(S, \Sigma)$ then $||\mu|| = |\mu|(S)$, the total variation of $\mu$ on $S$. The atomic part of $\mu$ is defined by $\nu(E) = \sum_{s \in E} \mu(s)$ where $E \subseteq \Sigma$. We shall need the facts that $C(\beta S)$ is isometrically isomorphic to $B(S)$ and $M(\beta S)$ is isometrically isomorphic to $ba(S, \Sigma)$. For a complete discussion of these facts see Dunford and Schwartz [3, p. 311–313].

Grothendieck's proof of Theorem 5 was based on the following result due to Phillips [7].

Theorem 6. Let $S$ be a discrete set and $\{\mu_n\}$ a sequence in $ba(S, \Sigma)$. If $\{\mu_n(E)\}$ converges to 0 for each $E \subseteq \Sigma$, then $\{||\nu_n||\}$ converges to 0, where $\nu_n$ is the atomic part of $\mu_n$.

Remark (Bade). Theorem 5 is equivalent to Theorem 6.

Proof. Assume Theorem 5 and that $\{\mu_n\}$ is a sequence in $ba(S, \Sigma)$ such that $\lim_n \mu_n(E) = 0$ for every $E \subseteq \Sigma$. Let $\bar{\mu}_n$ be the correspondent of $\mu_n$ in $M(\beta S)$, and $k_E$ be the correspondent of $k_E$ in $C(\beta S)$ ($k_E$ denotes the characteristic function of $E$). Then $\bar{\mu}_n(k_E) = \mu_n(k_E)$, and it follows that $\bar{\mu}_n$ converges to 0 for each open closed set in $\beta S$. By Theorem 5 $\{\bar{\mu}_n\}$ converges to 0 in the $\sigma(M(\beta S), M(\beta S)^*)$ topology. Thus $\{\mu_n\}$ converges to 0 for the $\sigma(ba(S, \Sigma), ba(S, \Sigma)^*)$ topology.

Let $P$ denote the projection of norm 1 of $ba(S, \Sigma)$ onto $l_1(S)$ defined
by $P: \mu \rightarrow \nu$ where $\nu$ is the atomic part of $\mu$. $P$ is norm continuous and hence is continuous for the weak topologies. Thus \( \{ P_{\mu_n} \} = \{ \nu_n \} \) converges to 0 for the $\sigma(l_1(S), l_\infty(S))$ topology. By a theorem of Banach [1, p. 137] \( \{ \| \nu_n \| \} \) converges to zero.

Assume Theorem 6 and that \( \{ \mu_n \} \) is a sequence of regular Borel measures on a Stonian space $S$ which converges to 0 on each open closed subset of $S$. To show that \( \{ \mu_n \} \) is weakly convergent to 0, it suffices to show (by Theorem 1) that \( \{ \mu_n(E_n) \} \) converges to 0, where \( \{ E_n \} \) is an arbitrary sequence of pairwise disjoint open closed subsets of $S$.

Define for each $n$ a set function $\nu_n$ on $N$, the set of positive integers:

\[
\nu_n(A) = \mu_n \left( \bigvee_{i \in A} E_i \right) \quad \text{where} \quad A \subseteq N.
\]

Note that $\nu_n$ is bounded and finitely additive, and hence an element of $ba(N, \Sigma)$. Since \( \{ \mu_n \} \) converges to 0 on each open closed subset of $S$, \( \{ \nu_n(A) \} \) converges to 0 for each $A \in \Sigma$. Theorem 6 allows the conclusion that $\lim_n \sum_{i=1}^{\infty} | \nu_n(i) | = 0$. In particular

\[
\lim_n | \nu_n(n) | = \lim_n | \mu_n(E_n) | = 0. \quad \text{Q.E.D.}
\]

The proof of Theorem 4 thus provides a new proof of Theorem 6.

References


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