A LOWER BOUND FOR THE DIMENSION OF CERTAIN G_δ SETS IN COMPLETELY NORMAL SPACES[1]

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Nagami and Roberts have proved [3, Theorem 1] that if X is a normal space of dimension at least n satisfying certain conditions, then $\dim (X - \bigcup_{i=1}^{\infty} A_i) \geqq n-1$ if the A_i are disjoint closed subsets of X. In this paper we allow the A_i to intersect provided that the dimension of the pairwise intersections is known. (The dimension of the remainder is reduced accordingly.)

By dimension (denoted dim) we mean the covering dimension. The symbols bdy and int are used to denote the topological boundary and interior. We will say that a T_1 space X is n-solid (n a positive integer) if for every point x of X and every open neighborhood U of x there is a connected open neighborhood V of x such that $\overline{V} \subseteq U$, $\overline{V}$ is compact, $\dim \overline{V} \geqq n$, and no $(n-2)$-dimensional subset of $\overline{V}$ separates it. Such a space is locally compact and locally connected. This property is hereditary on open subsets.

THEOREM 1. *Let m and n be integers such that $m \geqq -1$ and $n \geqq 1$. Let X be a compact completely normal space with* $\dim X \geqq n$. *Let A_1, $A_2, \cdots$ be a countable sequence of closed subsets of X such that*

(1) $\dim A_i \leqq n-1$ *for every integer i,*

(2) $\dim (A_i \cap A_j) \leqq m$ *if $i \neq j$.*

Then $\dim (X - \bigcup_{i=1}^{\infty} A_i) \geqq n-m-2$.

PROOF. The proof of the present theorem is similar to that of Theorem 1 in [3]. That proof has five steps:

(1) X is assumed to be a closed set with a certain minimal property.

(2) A function h from X into I^n is obtained.

(3) An n-dimensional pyramid is constructed in I^n.

(4) Assuming the theorem is false, the inverse images in X of the $(n-1)$ pairs of opposite faces of the pyramid are separated by $n-1$ closed sets whose intersection contains no continuum hitting the inverse images of both the base and the apex of the pyramid.

(5) The pyramid is truncated and the inverse images of the n pairs of opposite faces of the lower portion are shown not to be a defining

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system [3, Definition 5], in X. This is shown to be false, and a contradiction results.

The only change required in order to prove our present theorem is the substitution of the following modified step four:

Fourth step. Suppose that $\dim (X-\bigcup_{i=1}^{\infty} A_i)<n-m-2$. Then $\dim (X-(\bigcup_{i=1}^{\infty} A_i \cup h^{-1}(p)))$ is less than $n-m-2$ since $h^{-1}(p)$ is a G_δ. By Lemma 1 of [3] there exist sets $B_1, \cdots, B_{n-m-2}$ which are closed in $X-h^{-1}(p)$ and such that

(1) B_i separates C_i from C_i', $1 \leqq i \leqq n-m-2$,

(2) $\bigcap_{i=1}^{n-m-2} B_i \subseteq \bigcup_{i=1}^{\infty} A_i$.

Let $A=\bigcup_{i\neq j} (A_i \cap A_j)$; then A is an F_σ, and $\dim A \leqq m$. By [2, Theorem 3.4], there exist $m+1$ sets $B_{n-m-1}, \cdots, B_{n-1}$ which are closed in $X-h^{-1}(p)$ and such that

(1) B_i separates C_i from C_i', $n-m-1 \leqq i \leqq n-1$,

(2) $\bigcap_{i=n-m-1}^{n-1} B_i \subseteq X-A$.

Let $H=\bigcap_{i=1}^{n-1} B_i \subseteq \bigcup_{i=1}^{\infty} A_i - A$. $\overline{H}=H\cup h^{-1}(p)$ is compact, and $H\cap h^{-1}(B)$ and $h^{-1}(p)$ are disjoint closed subsets of $\overline{H}$. Recall from step 2 that $h^{-1}(p)\subseteq X-\bigcup_{i=1}^{\infty} A_i$. Then by the above $\overline{H}=h^{-1}(p) \cup (H\cap A_1)\cup(H\cap A_2)\cup \cdots$, and these pieces are disjoint and closed ($\overline{H}\cap A_i = H\cap A_i$). As in step 4 of [3, Theorem 1], there is no continuum $K\subseteq \overline{H}$ which intersects both $h^{-1}(p)$ and $H\cap h^{-1}(B)$.

Step 5 now proceeds unchanged, and the proof is complete.

Theorem 1 of [3] is obtained as a special case of this theorem when $m=-1$.

THEOREM 2. *If X is a connected n-solid space, then X is not the countable union of closed proper subsets $A_1, A_2, \cdots$ such that* $\dim (A_i \cap A_j) \leqq n-2$ *when $i \neq j$.*

PROOF. Suppose the theorem is false, and $X=\bigcup_{i=1}^{\infty} A_i$, where the A_i satisfy the conditions of the theorem. We will construct a sequence $\overline{V}_0, \overline{V}_1, \cdots$ of compact nonempty subsets of X such that for every integer i, $\overline{V}_{i+1} \subseteq \overline{V}_i$ and $\overline{V}_i \cap A_i = \emptyset$. This is a contradiction since the $\overline{V}_i$ have the finite intersection property and an empty total intersection.

By Baire's theorem there is an i such that int (A_i) is not empty. Fix i at this value. Then bdy $(\text{int}(A_i)) \neq \emptyset$ since X is connected. Let $x_0 \in$ bdy (int (A_i)). Let $x_0 \in V_0 \subseteq \overline{V}_0 \subseteq X$ be as in the definition of n-solid. Then $\overline{V}_0$ is not a subset of A_i since x_0 is not in int (A_i). Suppose there is an integer $j\neq i$ such that $\overline{V}_0 \subseteq A_j$. Then $A_j \cap A_i \supseteq V_0 \cap$ int (A_i), a nonempty open set. Hence $A_j \cap A_i$ contains a closed set of dimension at least n by the definition of n-solid, so $\dim (A_i \cap A_j)$

$\geqq n$, a contradiction. Then x_0 and V_0 satisfy (1)–(4) below if we let $V_{-1}=X$ and $A_0=\emptyset$.

Induction step. Suppose we are given x_p and V_p such that

(1) $x_p\in V_p$, and $\overline{V}_p$ is as in the definition of n-solid;

(2) for every integer j, $\overline{V}_p\not\subseteq A_j$;

(3) $\overline{V}_p\subseteq\overline{V}_{p-1}$;

(4) $\overline{V}_p\cap A_p=\emptyset$.

We will construct x_{p+1} and V_{p+1} satisfying (1)–(4).

Let W be the open set V_p-A_{p+1}. If W is connected, then it is not a subset of any A_j, since otherwise $\overline{V}_p\subseteq A_j\cup A_{p+1}$, so $A_j\cap A_{p+1}$ separates $\overline{V}_p$, a contradiction.

If W is not connected, let K be a component; K is open. Let $L=\operatorname{int}(\overline{K})$, so $K\subseteq L\subseteq\overline{L}=\overline{K}$. But some other component of W is not empty and V_p is connected. Therefore bdy $(L)\cap V_p$ is nonempty, separates V_p, and is a subset of A_{p+1}.

There is no integer j such that $K\subseteq A_j$, for suppose there is such a j. Then $\overline{K}\subseteq A_j$, so bdy $(L)\cap V_p\subseteq A_j\cap A_{p+1}$. Let $x\in$bdy $(L)\cap V_p$ and let $x\in V\subseteq\overline{V}\subseteq V_p$ be as in the definition of n-solid. Then V is not a subset of $\overline{L}$ since x is not in $L=\operatorname{int}\overline{L}$. Also $V\cap L\neq\emptyset$ since x is a limit point of L. Therefore bdy $(L)\cap\overline{V}$ separates $\overline{V}$, a contradiction.

In either case we get a connected open subset (K or W) of X which is not a subset of any A_j. Since this set (K or W) is n-solid, we can repeat the process used to get x_0 and V_0. This gives us the x_{p+1} and V_{p+1} we want, completing the induction and the proof.

It is well known that no continuum is a countable union of disjoint closed subsets. In particular, no locally connected continuum is such a union. Since a locally connected continuum is 1-solid, Theorem 2 is a direct generalization of this special case.

Nagami and Roberts have given an example [**3**, Figure 2] which shows that the conclusion of Theorem 2 does not hold if we require only that X be a Cantor n-manifold. This example is a Cantor 2-manifold which is the countable union of disjoint closed sets $A_{i,j}$ and a Cantor discontinuum.

COROLLARY. *If X is Hausdorff, connected, locally connected, and locally compact, then X is not a countable union of disjoint closed subsets.*

PROOF. Such a space is 1-solid.

Sierpinsky has given an example (see [**1**, p. 188]) which shows that local connectedness is not superfluous in the Corollary.

THEOREM 3. *Let m and n be integers such that $m\geqq-1$ and $n\geqq 1$. If X is connected, and every point in X has a neighborhood homeomorphic* *to the n-cube* I^n, *and* $A_1, A_2, \cdots$ *are closed proper subsets of* X *such that* $\dim (A_i \cap A_j) \leqq m$ *whenever* $i \neq j$, *then* $\dim (X - \bigcup_{i=1}^{\infty} A_i) \geqq n - m - 2$.

PROOF. X is connected and n-solid. By Theorem 2 there is a point $x \in X - \bigcup_{i=1}^{\infty} A_i$. There is a neighborhood F of x which is homeomorphic to I^n. The dimension of $X - \bigcup_{i=1}^{\infty} A_i$ is at least as great as the dimension of $F - \bigcup_{i=1}^{\infty} A_i$, so it is sufficient to prove the theorem for the case where $X = I^n$.

Let S^{n-1} denote the surface of I^n. Then $(I^n - S^{n-1})$ is connected and n-solid, and no A_i contains all of it. By Theorem 2 there is a point $q \in (I^n - S^{n-1}) - \bigcup_{i=1}^{\infty} A_i$.

Recall step 3 of the proof of Theorem 1 in [3]. A pyramid with apex p and base B has been inscribed in I^n. Let h be a homeomorphism from I^n onto I^n which leaves S^{n-1} fixed and such that $h(q) = p$. Then $h^{-1}(p) \cap (\bigcup_{i=1}^{\infty} A_i) = \varnothing$ as before. Let $f = h | S^{n-1}$, the identity map. Now, using these mappings h and f, repeat step 3, the modified step 4, and step 5 to complete the proof.

Corollary 2 of [3] is the special case of this theorem obtained when $m = -1$.

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