DETERMINANTS ON SEMILATTICES

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1. The following theorem contains Theorem 2 in [1] as a special case:

**Theorem.** Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a semilattice with \( \wedge \) as product and partially ordered such that \( a \leq b \) if and only if \( a \wedge b = a \). Given functions \( f_i(x), x \leq x_i(x), i = 1, 2, \ldots, n \), with values in a commutative ring with unit, the following equality holds:

\[
\det(f_i(x_i \wedge x_j))_{i,j=1}^n = \prod_{i=1}^n \sum_{j=1}^n f_i(x_j) \mu(x_j, x_i),
\]

where \( \mu \) is the Möbius function of \( X \).

Let \( f_i(x_j) = \text{sign} \mu(x_j, x_i) = 1 \) or \(-1\) for \( \mu(x_j, x_i) \geq 0 \) or \(<0\) resp., then we get as a corollary

**Corollary.**

\[
\det(\text{sign} \mu(x_i \wedge x_j, x_i))_{i,j=1}^n = \prod_{i=1}^n \sum_{j=1}^n | \mu(x_j, x_i) | .
\]

This corollary can be applied to the construction of some \((\pm 1)\)-determinants with large values.

For the background on generalized Möbius functions we refer to the paper [2] by Gian-Carlo Rota.

2. We first prove a lemma.

**Lemma.** Let \( X \) be a finite-semilattice and \( a, b \in X \) such that \( b \leq a \). Then for a function \( f(x), x \leq a \wedge b \), with values in a commutative ring with unit, we get

\[
S = \sum_{x \leq b} f(x \wedge a) \mu(x, b) = 0.
\]

**Proof.** By a theorem of L. Weisner [2, p. 351], \( \sum_{x \leq a \wedge b} \mu(x, 1) = 0 \) if \( a < 1 \). We find, since \( a \wedge b < b \),

\[
S = \sum_{y \in a \wedge b} f(y) \sum_{x \leq a \wedge y} \mu(x, b) = \sum_{y \in a \wedge b} f(y) \sum_{x \leq a \wedge b \leq y} \mu(x, b) = 0.
\]

The restriction \( x \leq b \) is omitted, since \( \mu(x, b) = 0 \) for \( x \not\leq b \).

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Proof of the Theorem. Permute indices for the elements in $X$ such that $x_i < x_j$ only if $i < j$. The value of the determinant is not changed when rows and columns are permuted in the same way. The matrix $M = (\mu(x_i, x_j))_{i,j=1}^n$ is now triangular with 0's below the main diagonal and 1's on it. Hence $\det M = 1$. Define the matrix $N = (f_i(x_i \wedge x_j))_{i,j=1}^n$. It follows from the lemma that the product $NM$ is triangular with all elements 0 above the main diagonal. We find

$$\det N = \det NM = \prod_{i=1}^n \sum_{j=1}^n f_i(x_j) \mu(x_j, x_i),$$

and the theorem is proved.

We give a few examples:

Example 1. Let $X$ be a (closed) family of finite sets such that $N \subseteq X$ and $M \subseteq N$ implies $M \subseteq X$. $X$ is then a semilattice with $\wedge$ as the operation of taking intersections. By the principle of inclusion-exclusion [2, p. 345], $\mu(M, N) = (-1)^{n(N) - n(M)}$ for $M \subseteq N$ ($n(N)$ is the cardinality of $N$). Theorem 2 in [1] follows easily.

Example 2. Let $X = \{1, 2, 3, 5, 30\}$ ordered by divisibility. From the Corollary we get a $(\pm 1)$-determinant of the order 5 with the value 48, which is maximum for $(\pm 1)$-determinants of the order 5 (cf. [3, p. 82]). Let $Y = \{1, 7\}$ and take the direct product of the lattices $X$ and $Y$. We get a lattice of the order 10 and then by the Corollary a $(\pm 1)$-determinant of the order 10 with the value $9 \cdot 2^{13}$, which is maximum for $(\pm 1)$-determinants of the order 10.

By trial one easily determines lattices of the orders 12 and 13 and corresponding $(\pm 1)$-determinants with values $> 1/3$ of the maximum for these determinants.

References


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