DETERMINANTS ON SEMILATTICES

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1. The following theorem contains Theorem 2 in [1] as a special case:

THEOREM. *Let* $X=\{x_1, x_2, \cdots, x_n\}$ *be a semilattice with* $\wedge$ *as product and partially ordered such that* $a \leqq b$ *if and only if* $a \wedge b = a$. *Given functions* $f_i(x)$, $x \leqq x_i(x)$, $i=1, 2, \cdots, n$, *with values in a commutative ring with unit, the following equality holds*:

$$\det(f_i(x_i \wedge x_j))_{i,j=1}^{n} = \prod_{i=1}^{n} \sum_{j=1}^{n} f_i(x_j)\mu(x_j, x_i),$$

where μ *is the Möbius function of* X.

Let $f_i(x_j) = \operatorname{sign} \mu(x_j, x_i) = 1$ or -1 for $\mu(x_j, x_i) \geqq 0$ or <0 resp., then we get as a corollary

COROLLARY.

$$\det(\operatorname{sign} \mu(x_i \wedge x_j, x_i))_{i,j=1}^{n} = \prod_{i=1}^{n} \sum_{j=1}^{n} | \mu(x_j, x_i) | .$$

This corollary can be applied to the construction of some (± 1)-determinants with large values.

For the background on generalized Möbius functions we refer to the paper [2] by Gian-Carlo Rota.

2. We first prove a lemma.

LEMMA. *Let* X *be a finite-semilattice and* $a, b \in X$ *such that* $b \nleqq a$. *Then for a function* $f(x)$, $x \leqq a \wedge b$, *with values in a commutative ring with unit, we get*

$$S = \sum_{x \leqq b} f(x \wedge a)\mu(x, b) = 0.$$

PROOF. By a theorem of L. Weisner [2, p. 351], $\sum_{x \wedge a = b} \mu(x, 1) = 0$ if $a < 1$. We find, since $a \wedge b < b$,

$$S = \sum_{y \leqq a \wedge b} f(y) \sum_{x \wedge a = y} \mu(x, b) = \sum_{y \leqq a \wedge b} f(y) \sum_{x \wedge a \wedge b = y} \mu(x, b) = 0.$$

The restriction $x \leqq b$ is omitted, since $\mu(x, b) = 0$ for $x \nleqq b$.

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PROOF OF THE THEOREM. Permute indices for the elements in X such that $x_i < x_j$ only if $i < j$. The value of the determinant is not changed when rows and columns are permuted in the same way. The matrix $M = (\mu(x_i, x_j))_{i,j=1}^n$ is now triangular with 0's below the main diagonal and 1's on it. Hence $\det M = 1$. Define the matrix $N = (f_i(x_i \wedge x_j))_{i,j=1}^n$. It follows from the lemma that the product NM is triangular with all elements 0 above the main diagonal. We find

$$\det N = \det NM = \prod_{i=1}^{n} \sum_{j=1}^{n} f_i(x_j)\mu(x_j, x_i),$$

and the theorem is proved.

We give a few examples:

EXAMPLE 1. Let X be a (closed) family of finite sets such that $N \in X$ and $M \subset N$ implies $M \in X$. X is then a semilattice with $\wedge$ as the operation of taking intersections. By the principle of inclusion-exclusion [2, p. 345], $\mu(M, N) = (-1)^{n(N)-n(M)}$ for $M \subset N$ ($n(N)$ is the cardinality of N). Theorem 2 in [1] follows easily.

EXAMPLE 2. Let $X = \{1, 2, 3, 5, 30\}$ ordered by divisibility. From the Corollary we get a (± 1)-determinant of the order 5 with the value 48, which is maximum for (± 1)-determinants of the order 5 (cf. [3, p. 82]). Let $Y = \{1, 7\}$ and take the direct product of the lattices X and Y. We get a lattice of the order 10 and then by the Corollary a (± 1)-determinant of the order 10 with the value $9 \cdot 2^{13}$, which is maximum for (± 1)-determinants of the order 10.

By trial one easily determines lattices of the orders 12 and 13 and corresponding (± 1)-determinants with values $> 1/3$ of the maximum for these determinants.

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