ELLIPTICITY AND REGULARITY FOR PERIODIC NONLINEAR EQUATIONS

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The composition of two strongly elliptic linear operators with suitably differentiable coefficients is also strongly elliptic. This fact has been used [**3**, pp. 178–181] to yield a particularly simple proof of the regularity of weak solutions of periodic strongly elliptic linear equations. In this paper we prove a similar regularity theorem for periodic nonlinear equations under the assumption of strong ellipticity for the composition of the given nonlinear operator with certain linear operators.

Let A be a nonlinear differential operator of order $2m$ given by

$$Au(x) = \sum_{|\alpha| \leq m} D^\alpha a_\alpha(x, \xi(u)(x)) \tag{1}$$

where $\xi = (\xi_\alpha)_{|\alpha| \leq m}$ and $\xi_\alpha(u) \equiv D^\alpha u$. Here, as usual, $x = (x_1, \cdots, x_n)$; $\alpha = (\alpha_1, \cdots, \alpha_n)$; $|\alpha| = \alpha_1 + \cdots + \alpha_n$; $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, the α_j being nonnegative integers and $D_j = i^{-1}\, \partial/\partial x_j$, $i^2 = -1$. Denote by P the class of all functions on R_n periodic with period unity in each variable. We assume throughout that $a_\alpha(\cdot, \xi) \in P$. If Q denotes a cube of unit edge with edges parallel to the coordinate axes in R_n we define the standard Dirichlet form for A:

$$a(u, v) = \sum_{|\alpha| \leq m} \int_Q a_\alpha(x, \xi(u)(x)) \overline{D^\alpha v(x)}\, dx, \qquad u, v \in P.$$

Let H_m denote the Hilbert space obtained by completing in the topology generated by the inner product

$$[u, v]_m = \sum_{|\alpha| \leq m} \int_Q D^\alpha u(x) \overline{D^\alpha v(x)}\, dx$$

the class $P \cap C^\infty(R_n)$ of all possibly complex-valued functions which are periodic with period unity in each coordinate and infinitely differentiable on R_n. We denote $\|u\|_m = (u, u)^{1/2}$. The dual of H_m with respect to the L_2 inner product $[u, v]_0$ is denoted H_{-m}. Properties of these spaces are discussed in [1, pp. 165–169]. If $f \in H_k$, then $D^\alpha f \in H_{k-|\alpha|}$ for any distribution derivative D^α. Given $f \in H_k$, $k \geqq -m$, u is said to be a periodic weak solution of the equation $Au = f$ if $u \in H_m$ and for each $v \in H_m$

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$$a(u, v) = \langle f, v\rangle,$$

where $\langle\cdot, \cdot\rangle$ denotes the pairing between H_{-m} and H_m.

The operator A will be called *elliptic of class* E_m if the following three conditions are satisfied:

(i) For each $r \geqq 0$ there exists a number $g(r)$ such that for all u, $v \in H_m$

$$|a(u, v)| \leqq g(\|u\|_m)\|v\|_m.$$

(ii) If $u_k \to u$ in H_m, then $a(u_k, v) \to a(u, v)$ for all v in H_m.

(iii) There exists a constant $c > 0$ such that for all u, $v \in H_m$

$$|a(u, u-v) - a(v, u-v)| \geqq c\|u-v\|_m^2.$$

For a given integer $k \geqq 0$, A will be called *elliptic of class* $E_{m,k}$ if for each integer j with $0 \leqq j \leqq k$ the operator $A_j = L^jA$ of order $2m+2j$ is elliptic of class E_{m+j} where $L = 1 - \Delta = 1 + \sum_{j=1}^n D_j^2$. (Note that if the coefficients a_α are sufficiently differentiable then A_j is an operator formally of type (1). If A is linear with C^∞ coefficients and is elliptic of class E_m, then it is also elliptic of class $E_{m,\infty}$.)

The existence of a periodic weak solution of $Au = f$ for $f \in H_k$, $k \geqq -m$ follows immediately for A elliptic of class E_m from the nonlinear extension of the Lax-Milgram theorem due to Zarantonello and Browder [2]. Specifically, (i) implies that there exists an operator T mapping H_m into itself such that $a(u, v) = [Tu, v]_m$ for all u, $v \in H_m$; (ii) shows that T is demicontinuous (continuous from the strong to the weak topology of H_m) and (iii) shows, by the above-mentioned theorem, that T is one-to-one onto H_m and has a continuous inverse. Also $|\langle f, v\rangle| \leqq \|f\|_{-m}\|v\|_m$ so that $\langle f, v\rangle = [f_0, v]_m$ where $f_0 \in H_m$ is the projection of f onto H_m. A weak solution of $Au = f$ is given by $u = T^{-1}f_0$.

Of more interest is the following regularity

THEOREM. *Let $u \in H_m$ be a periodic weak solution of $Au = f$ where $f \in H_k$, $k \geqq -m$. If there exists a constant λ such that the operator $A' = A + \lambda L^{m-1}$ is elliptic of class $E_{m,m+k}$, then $u \in H_{2m+k}$.*

PROOF. It is sufficient to prove the theorem for the case $\lambda = 0$ for, assuming this already done and noting that if u is a periodic weak solution of $Au = f$ it is also a periodic weak solution of $A'u = f'$ where $f' = f + \lambda L^{m-1}u$, we have $f' \in H_p$ where $p = \min(k, -m+2)$. Since A' satisfies the condition of the theorem with $\lambda = 0$ it follows that $u \in H_{2m+p}$, whence $f' \in H_q$ where $q = \min(k, -m+4)$. Hence $u \in H_{2m+q}$. Continuing in this way we obtain $u \in H_{2m+k}$.

We suppose therefore that $\lambda=0$ and A is elliptic of class $E_{m,m+k}$. As in the existence argument above there exist, for $0\leqq j\leqq m+k$, demicontinuous, continuously invertible operators T_j mapping H_{m+j} one-to-one onto itself such that for all $u, v\in H_{m+j}$

$$a_j(u, v) = [T_ju, v]_{m+j},$$

where a_j is the Dirichlet form of A_j. Similarly, by the Lax-Milgram theorem, there exist homeomorphisms V_j mapping H_{m+j} onto itself such that for all $u, v\in H_{m+j}$

$$l_j(u, v) = [V_ju, v]_{m+j}$$

where l_j is the Dirichlet form for L^{m+j}. Hence, for $0\leqq j\leqq m+k$, $S_j = v_j^{-1}T_j$ is a one-to-one mapping of H_{m+j} onto itself which satisfies for all $u, v\in H_{m+j}$

$$a_j(u, v) = l_j(S_ju, v).$$

On the other hand, by integrating by parts and using periodicity, we see that for all $\phi, \psi\in P\cap C^\infty(R_n)$,

$$a_j(\phi, \psi) = a_0(\phi, L^j\psi) = l_0(S_0\phi, L^j\psi) = l_j(S_0\phi, \psi),$$

whence $w=S_j\phi-S_0\phi$ is a periodic weak solution of the linear equation $L^{m+j}w=0$ and as such is the null element of H_{m+j}. By completion S_j is the restriction of S_0 to H_{m+j} and so S_0 maps H_{m+j} one-to-one onto itself.

Now since u is a periodic weak solution of $Au=f$, $f\in H_k$, it follows that $v=S_0u$ is a periodic weak solution of the linear equation $L^mv=f$ which by Friedrichs' theorem (periodic case) [**3**, pp. 178–181] belongs to H_{2m+k}. Since S_0 maps H_{2m+k} one-to-one onto itself $u\in H_{2m+k}$ as required.

References

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