ALMOST COMPLEX SUBMANIFOLDS OF THE SIX SPHERE

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In this note we prove the following theorem.

THEOREM. *With respect to the usual almost complex structure, S^6 has no 4-dimensional almost complex submanifolds.*

The proof, which is entirely local, is divided into several lemmas. The usual almost complex structure J on S^6 can be defined either by means of the Cayley numbers [1], or by using the fact that $S^6 = G_2/SU(3)$. Let $\overline{\nabla}$ denote the Riemannian connection of S^6 with respect to the ordinary metric $\langle\ ,\ \rangle$, and let $\overline{R}$ be the curvature operator.

LEMMA 1. *If X is a vector field on S^6, then $\overline{\nabla}_X(J)(X) = 0$.*

The proof is given in [1]. Almost complex manifolds having this property are called *nearly Kählerian*. In [2] it is shown that S^6 has some almost complex structures, defined by means of a 3-fold vector cross product on R^8, which are different from the usual almost complex structure, but nonetheless nearly Kählerian. Our theorem will apply to these almost complex structures, also.

We next give an account of our machinery for describing the geometry of submanifolds. Let M and $\overline{M}$ be C^∞ Riemannian manifolds with M isometrically embedded in $\overline{M}$. Let $\mathfrak{X}(M)$ and $\overline{\mathfrak{X}}(M)$ denote respectively the Lie algebras of vector fields on M and the restrictions to M of vector fields on $\overline{M}$. Then we may write $\overline{\mathfrak{X}}(M) = \mathfrak{X}(M) \oplus \mathfrak{X}(M)^\perp$. The configuration tensor is the function $T: \mathfrak{X}(M) \times \overline{\mathfrak{X}}(M) \rightarrow \overline{\mathfrak{X}}(M)$ defined by the formulas

$$T_X Y = \overline{\nabla}_X Y - \nabla_X Y, \qquad T_X Z = P\overline{\nabla}_X Z,$$

for X, $Y \in \mathfrak{X}(M)$, $Z \in \mathfrak{X}(M)^\perp$. Here ∇ and $\overline{\nabla}$ are the Riemannian connections of M and $\overline{M}$ and $P: \overline{\mathfrak{X}}(M) \rightarrow \mathfrak{X}(M)$ is the natural projection. Then [1] T has the following properties:

$$T_X Y = T_Y X, \qquad \langle T_X Y, Z\rangle = -\langle T_X Z, Y\rangle,$$

$$T_X(\mathfrak{X}(M)) \subseteq \mathfrak{X}(M)^\perp, \qquad T_X(\mathfrak{X}(M)^\perp) \subseteq \mathfrak{X}(M).$$

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LEMMA 2. *Let $\overline{M}$ be a nearly Kähler manifold with almost complex structure J, and suppose M is an almost complex submanifold of $\overline{M}$* (i.e., *$X \in \mathfrak{X}(M)$ implies $JX \in \mathfrak{X}(M)$*). *Then*

(i) *M is nearly Kählerian;*

(ii) *$T_X JX = JT_X X$ for all $X \in \mathfrak{X}(M)$;*

(iii) *M is a minimal variety of $\overline{M}$.*

The proof is given in [1], but we include it here for the convenience of the reader.

PROOF. For $X \in \mathfrak{X}(M)$ we have

$$0 = \overline{\nabla}_X(J)(X) = \nabla_X(J)(X) + T_X JX - JT_X X.$$

The first term on the right-hand side of this equation is tangent to M, while the last is normal. Hence (i) and (ii) follow. For (iii) we have by (ii) that

$$T_X X = -JT_X JX = -T_{JX} JX.$$

Then the mean curvature vector $H = \sum T_{E_i} E_i$ vanishes (here $\{E_1, \cdots, E_n\}$ is any orthonormal frame field on an open subset of M), and so M is a minimal variety.

LEMMA 3. *A 4-dimensional nearly Kähler manifold M is Kählerian.*

PROOF. Choose an orthonormal frame field on an open subset of M to be of the form $\{X, JX, Y, JY\}$. We must show $\nabla_X(J)(Y) = 0$. Clearly $\nabla_X(J)(Y)$ is perpendicular to X and Y. It is also perpendicular to JX and JY because $\nabla_X(J)(Y) = J\nabla_X(J)(JY)$. Hence it vanishes.

LEMMA 4. *Let M be a nearly Kähler manifold and denote the curvature operator of M by R. Then for $W, X, Y, Z \in \mathfrak{X}(M)$, we have*

(i) $\| \nabla_X(J)(Y)\|^2 = \langle R_{XY}X, Y\rangle - \langle R_{XY}JX, JY\rangle$;

(ii) $\langle R_{WX}Y, Z\rangle = \langle R_{JWJX}JY, JZ\rangle$.

PROOF. Let F be the 2-form defined by $F(X, Y) = \langle JX, Y\rangle$ for $X, Y \in \mathfrak{X}(M)$. We also define ∇F, $\nabla^2 F$, and $R_{WX}(F)$ by

$$\nabla F(X, Y, Z) = XF(Y, Z) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z),$$

$$\begin{aligned}\nabla^2 F(W, X, Y, Z) = {} & W\nabla F(X, Y, Z) - \nabla F(\nabla_W X, Y, Z) \\ & - \nabla F(X, \nabla_W Y, Z) - \nabla F(X, Y, \nabla_W Z),\end{aligned}$$

and $R_{WX}(F)(Y, Z) = -F(R_{WX}Y, Z) - F(Y, R_{WX}Z)$, for $W, X, Y, Z \in \mathfrak{X}(M)$.

Calculations show first that

$$R_{WX}(F)(Y, Z) = -\nabla^2 F(W, X, Y, Z) + \nabla^2 F(X, W, Y, Z),$$

and secondly $\|\nabla_X(J)(Y)\|^2 = -\nabla^2F(X, X, Y, JY)$. These computations do not depend on the fact that M is nearly Kählerian. However, the fact that M is nearly Kählerian is used to prove that

$$\nabla^2F(W, X, Y, Z) = -\nabla^2F(W, Y, X, Z) \quad \text{for } W, X, Y, Z \in \mathfrak{X}(M).$$

We now observe that

$$\begin{aligned}\|\nabla_X(J)(Y)\|^2 &= \nabla^2F(X, Y, X, JY)\\ &= -R_{XY}(F)(X, JY)\\ &= \langle R_{XY}X, Y\rangle - \langle R_{XY}JX, JY\rangle,\end{aligned}$$

proving (i). Then (ii) follows from (i) and the symmetry properties of the curvature operator.

Let M_p and $\overline{M}_p$ denote the tangent spaces to M and $\overline{M}$ at a point $p\in M$. The configuration tensor T, the curvature operator $\overline{R}$, and the covariant derivative $\overline{\nabla}J$ all give rise to tensors on M_p or $\overline{M}_p$. For $x, y\in M_p$ we denote these by T_xy, $\overline{R}_xy$, $\overline{\nabla}_x(J)(y)$.

Proof of the Theorem. Suppose that M is a 4-dimensional almost complex submanifold of S^6. First assume that there exists $p\in M$ and $x\in M_p$ with $\|x\|=1$ such that $T_xx\neq 0$. Then $T_xJx=JT_xx\neq 0$. We may assume that x is chosen so that $\|T_xx\|^2$ is maximum. Then if $\langle x, y\rangle=0$ and $\langle Jx, y\rangle=0$, it is not hard to see that $\langle T_xx, T_xy\rangle=0$. Moreover, $\langle JT_xx, T_xy\rangle=0$ because $JT_xx=T_uu$ with $u=(x+Jx)/\sqrt{2}$. Similarly T_xJy is perpendicular to both T_xx and JT_xx. Since T_xx and JT_xx span $M_p^\perp$, we must have $T_xy=T_xJy=0$.

Let $\overline{K}$ denote the (constant) sectional curvature of the curvature operator $\overline{R}$ of S^6. Then if $\|y\|=1$, we have

$$\begin{aligned}\overline{K} &= \langle\overline{R}_{xy}x, y\rangle - \langle\overline{R}_{xy}Jx, Jy\rangle = \|\overline{\nabla}_x(J)(y)\|^2\\ &= \|\nabla_x(J)(y) + T_xJy - JT_xy\|^2 = 0.\end{aligned}$$

This is a contradiction, and so we conclude that $T_xx=0$ for all $x\in M_p$.

Hence M is totally geodesic in S^6, and so it must be an open submanifold of a 4-sphere with constant curvature $\overline{K}$, and also must be Kählerian. We then have

$$0 = \|\nabla_x(J)(y)\|^2 = \overline{K},$$

another contradiction. Hence the theorem follows.

Bibliography

1. A. Gray, *Minimal varieties and almost Hermitian submanifolds*, Michigan Math. J. **12** (1965), 273–287.

2. ———, *Vector cross products on manifolds*, (to appear).

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