ON A THEOREM OF FEJÉR AND RIESZ

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1. In what follows we suppose that \( r > 1 \) and that \( A_r \) is a constant, depending only on \( r \), the value of which is not usually the same at each occurrence. Let \( U(\theta) \) denote a real function measurable over \((-\pi, \pi)\), let

\[ P(p, \theta) = \frac{1}{(1 - p^2)^2 + 4p \sin^2 \frac{\theta}{2}}, \quad 0 \leq p < 1, \]

and let

\[ u(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(p, \theta) U(\theta) d\theta. \]

\( P(p, \theta) \) is the Poisson kernel and \( u(p) \equiv u(p, 0) \) is the value at the point \((p, 0)\) of a function \( u(p, \theta) \) in polar coordinates which is harmonic inside the unit disc and has boundary value \( U(\theta) \) on the unit circle.

We begin by giving a new 'real variable' proof of the following theorem of Fejér-Riesz type. This is similar to a proof given by du Plessis [3] but it differs in a way which leads to a new analogue in three dimensions. Before du Plessis' paper appeared, the only proof available was of a 'complex variable' nature and based on the Fejér-Riesz inequality \( \int_0^1 |f(r)| dr \leq \frac{1}{\pi} \int_0^{2\pi} |f(e^{i\theta})| d\theta \) [1].

**Theorem 1.**

\[(1) \quad \int_0^1 |u(p)| dp \leq A_r \int_{-\pi}^{\pi} |U(\theta)| d\theta. \]

The proof is based on the use of an inequality theorem (see, for example, [2, p. 229, Theorem 319]) which we state as a lemma.

**Lemma.** If \( f(x) \) is nonnegative, \( K(x, y) \) nonnegative and homogeneous of degree \(-1\) and \( \int_0^\infty K(x, 1)x^{-1/2}dx = k \), then

\[(2) \quad \int_0^\infty \left( \int_0^\infty K(x, y)f(x)dx \right) dy \leq k^r \int_0^\infty (f(x))^r dx. \]

**Proof of Theorem 1.** For \( 0 \leq \rho < 1 \), since \( P(\rho, \theta) > 0 \), we have

\[ |u(\rho)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P(\rho, \theta) |U(\theta)| d\theta, \]

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so it is enough to prove the theorem under the assumption that 
\[ U(\theta) \geq 0. \] We may further assume that \( U(\theta) \) is an even function of \( \theta \)
and prove (what is then equivalent to (1)) that

\[
\int_0^1 \left( \frac{1}{\pi} \int_0^\pi P(\rho, \theta) U(\theta) d\theta \right)^r d\rho \leq A_r \int_0^\pi (U(\theta))^r d\theta,
\]

for on replacing \( U(\theta) \) in (3) by the even function \( U(\theta) + U(-\theta) \), and
using the Hölder inequality \( (a+b)^r \leq 2^{r-1}(a^r+b^r) \) for \( a \geq 0, b \geq 0 \), we obtain

\[
\int_0^1 |u(\rho)|^r d\theta = \int_0^1 \left[ \frac{1}{2\pi} \int_0^\pi P(\rho, \theta)(U(\theta) + U(-\theta)) d\theta \right]^r d\rho \\
\leq A_r \int_0^\pi (U(\theta) + U(-\theta))^r d\theta \\
\leq A_r \int_0^\pi 2^{r-1}[(U(\theta))^r + (U(-\theta))^r] d\theta \\
= A_r \int_{-\pi}^\pi (U(\theta))^r d\theta.
\]

Dividing the range of integration with respect to \( \rho \) in (3) into the
two intervals \((0, \frac{1}{2}), (\frac{1}{2}, 1)\), we first consider integration over \((0, \frac{1}{2})\). Since \( P(\rho, \theta) \leq (1+\rho)/(1-\rho) \) we have, using Hölder's inequality,

\[
\int_0^{1/2} \left( \frac{1}{\pi} \int_0^\pi P(\rho, \theta) U(\theta) d\theta \right)^r d\rho \leq \int_0^{1/2} \pi^{-r} \left( \frac{1+\rho}{1-\rho} \right)^r \left( \int_0^\pi U(\theta) d\theta \right)^r d\rho \\
= A_r \left( \int_0^\pi U(\theta) d\theta \right)^r \\
\leq A_r \int_0^\pi (U(\theta))^r d\theta.
\]

Next, defining

\[
P_1(\rho, \theta) = \frac{2(1-\rho)}{((1-\rho)^2 + \theta^2)},
\]

for the interval \((\frac{1}{2}, 1)\), since \( \frac{1}{2} \theta \geq \theta/\pi \) over \((0, \pi)\), we have

\[
P(\rho, \theta) \leq \frac{2(1-\rho)}{(1-\rho)^2 + 2\theta^2/\pi^2} = \frac{1}{2} \pi^2 \frac{2(1-\rho)}{\frac{1}{2} \pi^2 (1-\rho)^2 + \theta^2} < \frac{1}{2} \pi^2 P_1(\rho, \theta),
\]

and so
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\[
\int_{1/2}^{1} \left( \frac{1}{\pi} \int_{0}^{\pi} P(\rho, \theta) U(\theta) d\theta \right)^r d\rho \leq A_r \int_{1/2}^{1} \left( \frac{1}{\pi} \int_{0}^{\pi} P_1(\rho, \theta) U(\theta) d\theta \right)^r d\rho.
\]

With \( f(x) = U(x) \) for \( 0 \leq x \leq \pi, f(x) = 0 \) for \( x > \pi \), and
\[
K(x, y) = \frac{2}{\pi} y/(x^2 + y^2),
\]
the functions \( f(x), K(x, y) \) satisfy the conditions of the lemma (with \( k = \csc \pi(1 - 1/r) \)). On replacing \( x \) by \( \theta, y \) by \( 1 - \rho \), (2) and (5) give
\[
\int_{1/2}^{1} \left( \frac{1}{\pi} \int_{0}^{\pi} P_1(\rho, \theta) U(\theta) d\theta \right)^r d\rho \leq A_r \int_{0}^{\pi} (U(\theta))^r d\theta,
\]
and finally (4) and combination of (6) and (7) give the inequality (3). This completes the proof of the theorem.

2. When analogues of Theorem 1 for functions harmonic in the unit sphere are considered there are two possibilities. Let \((\rho, \theta, \phi)\) denote spherical polar coordinates, \(U(\theta, \phi)\) a real function measurable for \(0 \leq \theta \leq \pi, -\pi \leq \phi \leq \pi\), and let
\[
Q(\rho, \theta) = (1 - \rho^2)^{-3/2} \sin \theta,
\]
\[
u(\rho) = \int_{0}^{\pi} \left( \int_{-\pi}^{\pi} U(\theta, \phi) d\phi \right) Q(\rho, \theta) d\theta.
\]
Here \(Q(\rho, \theta)\) is the three-dimensional Poisson kernel and \(u(\rho) = u(\rho, 0, 0)\) is the value at the point \((\rho, 0, 0)\) of a function \(u(\rho, \theta, \phi)\) harmonic inside the unit sphere and with boundary values \(U(\theta, \phi)\) on the surface. The possibilities are
\[
\int_{0}^{1} \left( \int_{0}^{\rho} |u(\rho)| \rho d\rho \right) d\rho \leq A_r \int_{0}^{\pi} \int_{-\pi}^{\pi} |U(\theta, \phi)| \rho \sin \theta d\phi d\theta.
\]
and
\[
\int_{0}^{1} |u(\rho, \theta, \phi)| \rho d\phi d\theta \leq A_r \int_{0}^{\pi} \int_{-\pi}^{\pi} |U(\theta, \phi)| \rho \sin \theta d\phi d\theta.
\]
In both inequalities the right-hand side is the integral of \( |U(\theta, \phi)| \rho \) over the surface of the unit sphere. In the first inequality, the left-hand side consists of two integrations of \(u(\rho, \theta, \phi)\) along a radius, in the second inequality the left-hand side is the integral of \(u(\rho, \theta, \phi)\) over a diametral plane. Both of these analogues are, in fact, valid and
they are particular cases of a general theorem of du Plessis [3] concerning functions in \( n \) dimensions. du Plessis' proof of this general theorem is indirect and depends on half-space analogues of Theorem 1 [4]. In this note we give a direct proof of a stronger version of (8) which does not seem to be deducible using du Plessis' argument.

**Theorem 2.**

(9) \[ \int_0^1 (1 - \rho) \left| \frac{\partial}{\partial \rho} \right| u(\rho) \, d\rho \leq A_r \int_0^\pi \left( \int_{-\pi}^\pi \left| U(\theta, \phi) \right| \, d\phi \right)^r \sin \theta d\theta. \]

The left-hand side here is identical to the left-hand side of (8) and, by Hölder's inequality,

\[ \int_0^\pi \left( \int_{-\pi}^\pi \left| U(\theta, \phi) \right| \, d\phi \right)^r \sin \theta d\theta \leq A_r \int_0^\pi \int_{-\pi}^\pi \left| U(\theta, \phi) \right|^r \sin \theta d\phi d\theta, \]

so that (9) is a stronger inequality than (8).

**Proof.** Arguing as before, it is enough to prove the theorem under the assumption that \( U(\theta, \phi) \geq 0 \).

We divide the range of integration with respect to \( \rho \) as before, and first consider integration over \((0, \frac{1}{2})\). Since

\[ Q(\rho, \theta) \leq (1 + \rho)(1 - \rho)^{-2} \sin \theta \]

we have, using Hölder's inequality,

\[ \int_0^{1/2} (1 - \rho)(u(\rho))^r d\rho \]

\[ = \int_0^{1/2} (1 - \rho) \left[ \frac{1}{4\pi} \int_0^\pi \left( \int_{-\pi}^\pi U(\theta, \phi) \, d\phi \right) Q(\rho, \theta) \, d\theta \right]^r \, d\rho \]

\[ \leq \int_0^{1/2} (1 - \rho)(4\pi)^{-r} \frac{(1 + \rho)^r}{(1 - \rho)^{2r}} \]

(10) \[ \left[ \int_0^\pi \left( \int_{-\pi}^\pi U(\theta, \phi) \, d\phi \right) \sin \theta d\theta \right]^r \, d\rho \]

\[ = A_r \left[ \int_0^\pi \left( \int_{-\pi}^\pi U(\theta, \phi) \, d\phi \right) \sin \theta d\theta \right]^r \]

\[ \leq A_r \int_0^\pi \left( \int_{-\pi}^\pi U(\theta, \phi) \, d\phi \right)^r \sin \theta d\theta \]

\[ \leq A_r \int_0^\pi \left( \int_{-\pi}^\pi U(\theta, \phi) \, d\phi \right)^r \sin \theta d\theta. \]
Next, defining

$$R(p, \theta) = 2(1 - \rho)^{1 + 1/\rho - 1/\rho}[(1 - \rho)^2 + \theta^2]^{-3/2}$$

for the interval $\left(\frac{1}{2}, 1\right)$, since $\sin \theta \leq \theta$ and $\frac{1}{2} \theta \leq \theta/\pi$ over $(0, \pi)$, we have

$$Q(p, \theta) \leq 2(1 - \rho)[(1 - \rho)^2 + 2\theta^2/\pi^2]^{-3/2} \sin \theta$$

$$= 2^{-1/2} \pi^3(1 - \rho)[(1 - \rho)^2 + \theta^2]^{-3/2} \sin \theta$$

$$\leq 2^{-1/2} \pi^3(1 - \rho)^{-1/\rho} \sin^{1/\rho} \theta R(p, \theta),$$

and so

$$\int_{1/2}^1 (1 - \rho)(u(\rho))^r d\rho$$

$$= \int_{1/2}^1 (1 - \rho) \left[ \frac{1}{4\pi} \int_0^\pi \left( \int_{-\pi}^{\pi} U(\theta, \phi) d\phi \right) Q(p, \theta) d\theta \right] d\rho$$

$$\leq A_r \int_{1/2}^1 \left[ \frac{1}{4\pi} \int_0^\pi R(p, \theta) \left( \int_{-\pi}^{\pi} U(\theta, \phi) d\phi \right) \sin^{1/\rho} \theta \right] d\rho.$$

Defining $f(x) = \sin^{1/r} x \int_{-\pi}^{\pi} U(x, \phi) d\phi$ for $0 \leq x \leq \pi$, $f(x) = 0$ for $x > \pi$, and

$$K(x, y) = (2\pi)^{-1} x^{-1/\rho} y^{1 + 1/\rho} (x^2 + y^2)^{-3/2},$$

the functions $f(x)$, $K(x, y)$ satisfy the conditions of the lemma (with $k = \frac{1}{2} \pi^{-3/2} \Gamma(1 - 1/r) \Gamma(1/2 + 1/r)$). On replacing $x$ by $\theta$, $y$ by $1 - \rho$, (2) and (11) give

$$\int_{1/2}^1 \left[ \frac{1}{4\pi} \int_0^\pi R(p, \theta) \left( \int_{-\pi}^{\pi} U(\theta, \phi) d\phi \right) \sin^{1/\rho} \theta \right] d\rho$$

$$\leq A_r \int_0^\pi \left( \int_{-\pi}^{\pi} U(\theta, \phi) d\phi \right) \sin \theta d\theta.$$

Combination of (12) and (13) now gives

$$\int_{1/2}^1 (1 - \rho)(u(\rho))^r d\rho \leq A_r \int_0^\pi \left( \int_{-\pi}^{\pi} U(\theta, \phi) d\phi \right) \sin \theta d\theta,$$

and addition of (10) and (14) yields the desired inequality (9).
REFERENCES


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