ON A THEOREM OF FEJÉR AND RIESZ

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1. In what follows we suppose that $r > 1$ and that A_r is a constant, depending only on r, the value of which is not usually the same at each occurrence. Let $U(\theta)$ denote a real function measurable over $(-\pi, \pi)$, let

$$P(\rho, \theta) = (1 - \rho^2)[(1 - \rho)^2 + 4\rho \sin^2 \tfrac{1}{2}\theta]^{-1}, \qquad 0 \leqq \rho < 1,$$

and let

$$u(\rho) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\rho, \theta) U(\theta) d\theta.$$

$P(\rho, \theta)$ is the Poisson kernel and $u(\rho) \equiv u(\rho, 0)$ is the value at the point $(\rho, 0)$ of a function $u(\rho, \theta)$ in polar coordinates which is harmonic inside the unit disc and has boundary value $U(\theta)$ on the unit circle.

We begin by giving a new 'real variable' proof of the following theorem of Fejér-Riesz type. This is similar to a proof given by du Plessis [**3**] but it differs in a way which leads to a new analogue in three dimensions. Before du Plessis' paper appeared, the only proof available was of a 'complex variable' nature and based on the Fejér-Riesz inequality $\int_0^1 |f(r)| dr \leqq \frac{1}{2}\int_{-\pi}^{\pi} |f(e^{i\theta})| d\theta$ [**1**].

THEOREM 1.

$$\int_0^1 | u(\rho) |^r d\rho \leqq A_r \int_{-\pi}^{\pi} | U(\theta) |^r d\theta. \tag{1}$$

The proof is based on the use of an inequality theorem (see, for example, [**2**, p. 229, Theorem 319]) which we state as a lemma.

LEMMA. *If $f(x)$ is nonnegative, $K(x, y)$ nonnegative and homogeneous of degree -1 and $\int_0^\infty K(x, 1)x^{-1/r}dx = k$, then*

$$\int_0^\infty \left(\int_0^\infty K(x, y)f(x)dx\right)^r dy \leqq k^r \int_0^\infty (f(x))^r dx. \tag{2}$$

PROOF OF THEOREM 1. For $0 \leqq \rho < 1$, since $P(\rho, \theta) > 0$, we have

$$| u(\rho) | \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} P(\rho, \theta) | U(\theta) | d\theta,$$

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so it is enough to prove the theorem under the assumption that $U(\theta)\geqq 0$. We may further assume that $U(\theta)$ is an even function of θ and prove (what is then equivalent to (1)) that

$$\int_0^1\left(\frac{1}{\pi}\int_0^\pi P(\rho,\theta)U(\theta)d\theta\right)^r d\rho \leqq A_r\int_0^\pi (U(\theta))^r d\theta, \tag{3}$$

for on replacing $U(\theta)$ in (3) by the even function $U(\theta)+U(-\theta)$, and using the Hölder inequality $(a+b)^r\leqq 2^{r-1}(a^r+b^r)$ for $a\geqq 0$, $b\geqq 0$, we obtain

$$\begin{aligned}\int_0^1 |u(\rho)|^r d\theta &= \int_0^1\left[\frac{1}{2\pi}\int_0^\pi P(\rho,\theta)(U(\theta)+U(-\theta))d\theta\right]^r d\rho\\ &\leqq A_r\int_0^\pi (U(\theta)+U(-\theta))^r d\rho\\ &\leqq A_r\int_0^\pi 2^{r-1}[(U(\theta))^r+(U(-\theta))^r]d\theta\\ &= A_r\int_{-\pi}^\pi (U(\theta))^r d\theta.\end{aligned}$$

Dividing the range of integration with respect to ρ in (3) into the two intervals $(0,\frac{1}{2})$, $(\frac{1}{2},1)$, we first consider integration over $(0,\frac{1}{2})$. Since $P(\rho,\theta)\leqq(1+\rho)/(1-\rho)$ we have, using Hölder's inequality,

$$\begin{aligned}\int_0^{1/2}\left(\frac{1}{\pi}\int_0^\pi P(\rho,\theta)U(\theta)d\theta\right)^r d\rho &\leqq \int_0^{1/2}\pi^{-r}\left(\frac{1+\rho}{1-\rho}\right)^r\left(\int_0^\pi U(\theta)d\theta\right)^r d\rho\\ &= A_r\left(\int_0^\pi U(\theta)d\theta\right)^r\\ &\leqq A_r\int_0^\pi (U(\theta))^r d\theta.\end{aligned} \tag{4}$$

Next, defining

$$P_1(\rho,\theta) = \frac{2(1-\rho)}{((1-\rho)^2+\theta^2)}, \tag{5}$$

for the interval $(\frac{1}{2},1)$, since $\frac{1}{2}\theta\geqq\theta/\pi$ over $(0,\pi)$, we have

$$P(\rho,\theta)\leqq\frac{2(1-\rho)}{(1-\rho)^2+2\theta^2/\pi^2}=\frac{1}{2}\pi^2\frac{2(1-\rho)}{\frac{1}{2}\pi^2(1-\rho)^2+\theta^2}<\frac{1}{2}\pi^2P_1(\rho.\theta),$$

and so

$$\int_{1/2}^{1}\left(\frac{1}{\pi}\int_{0}^{\pi} P(\rho,\theta)U(\theta)d\theta\right)^{r} d\rho \tag{6}$$
$$\leqq A_r\int_{1/2}^{1}\left(\frac{1}{\pi}\int_{0}^{\pi} P_1(\rho,\theta)U(\theta)d\theta\right)^{r} d\rho.$$

With $f(x) = U(x)$ for $0 \leqq x \leqq \pi$, $f(x) = 0$ for $x > \pi$, and

$$K(x, y) = (2/\pi)y/(x^2 + y^2),$$

the functions $f(x)$, $K(x, y)$ satisfy the conditions of the lemma (with $k = \operatorname{cosec} \frac{1}{2}\pi(1 - 1/r)$). On replacing x by θ, y by $1-\rho$, (2) and (5) give

$$\int_{1/2}^{1}\left(\frac{1}{\pi}\int_{0}^{\pi} P_1(\rho,\theta)U(\theta)d\theta\right)^{r} d\rho \leqq \int_{-\infty}^{1} \leqq A_r\int_{0}^{\pi}(U(\theta))^r d\theta, \tag{7}$$

and finally (4) and combination of (6) and (7) give the inequality (3). This completes the proof of the theorem.

2. When analogues of Theorem 1 for functions harmonic in the unit sphere are considered there are two possibilities. Let (ρ, θ, ϕ) denote spherical polar coordinates, $U(\theta, \phi)$ a real function measurable for $0 \leqq \theta \leqq \pi$, $-\pi \leqq \phi \leqq \pi$, and let

$$Q(\rho, \theta) = (1 - \rho^2)[(1 - \rho)^2 + 4\rho \sin^2 \tfrac{1}{2}\theta]^{-3/2} \sin \theta,$$

$$u(\rho) = \int_{0}^{\pi}\left(\int_{-\pi}^{\pi} U(\theta,\phi)d\phi\right)Q(\rho,\theta)d\theta.$$

Here $Q(\rho, \theta)$ is the three-dimensional Poisson kernel and $u(\rho) \equiv u(\rho, 0, 0)$ is the value at the point $(\rho, 0, 0)$ of a function $u(\rho, \theta, \phi)$ harmonic inside the unit sphere and with boundary values $U(\theta, \phi)$ on the surface. The possibilities are

$$\int_{0}^{1}\left(\int_{0}^{\rho} | u(r) |^r dt\right) d\rho \leqq A_r \int_{0}^{\pi}\int_{-\pi}^{\pi} | U(\theta, \phi) |^r \sin \theta d\phi d\theta. \tag{8}$$

and

$$\int_{0}^{1} | u(\rho, \theta, \phi) |^r \rho d\phi d\theta \leqq A_r \int_{0}^{\pi}\int_{-\pi}^{\pi} | U(\theta, \phi) |^r \sin \theta d\phi d\theta.$$

In both inequalities the right-hand side is the integral of $|U(\theta, \phi)|^r$ over the surface of the unit sphere. In the first inequality, the left-hand side consists of two integrations of $u(\rho, \theta, \phi)$ along a radius, in the second inequality the left-hand side is the integral of $u(\rho, \theta, \phi)$ over a diametral plane. Both of these analogues are, in fact, valid and they are particular cases of a general theorem of du Plessis [**3**] concerning functions in n dimensions. du Plessis' proof of this general theorem is indirect and depends on half-space analogues of Theorem 1 [**4**]. In this note we give a direct proof of a stronger version of (8) which does not seem to be deducible using du Plessis' argument.

THEOREM 2.

$$\int_0^1 (1-\rho)\,|\,u(\rho)\,|^r d\rho \leqq A_r \int_0^\pi \left(\int_{-\pi}^{\pi} |\,U(\theta,\phi)\,|\,d\phi\right)^r \sin\theta d\theta. \tag{9}$$

The left-hand side here is identical to the left-hand side of (8) and, by Hölder's inequality,

$$\int_0^\pi \left(\int_{-\pi}^{\pi} |\,U(\theta,\phi)\,|\,d\phi\right)^r \sin\theta d\theta \leqq A_r \int_0^\pi \int_{-\pi}^{\pi} |\,U(\theta,\phi)\,|^r \sin\theta d\phi d\theta,$$

so that (9) is a stronger inequality than (8).

PROOF. Arguing as before, it is enough to prove the theorem under the assumption that $U(\theta,\phi)\geqq 0$.

We divide the range of integration with respect to ρ as before, and first consider integration over $(0,\frac{1}{2})$. Since

$$Q(\rho,\theta)\leqq(1+\rho)(1-\rho)^{-2}\sin\theta$$

we have, using Hölder's inequality,

$$\begin{aligned}
&\int_0^{1/2}(1-\rho)(u(\rho))^r d\rho \\
&\quad= \int_0^{1/2}(1-\rho)\left[\frac{1}{4\pi}\int_0^\pi\left(\int_{-\pi}^{\pi} U(\theta,\phi)d\phi\right)Q(\rho,\theta)d\theta\right]^r d\rho \\
&\quad\leqq \int_0^{1/2}(1-\rho)(4\pi)^{-r}\frac{(1+\rho)^r}{(1-\rho)^{2r}} \\
&\qquad\cdot\left[\int_0^\pi\left(\int_{-\pi}^{\pi} U(\theta,\phi)d\phi\right)\sin\theta d\theta\right]^r d\rho \qquad (10)\\
&\quad= A_r\left[\int_0^\pi\left(\int_{-\pi}^{\pi} U(\theta,\phi)d\phi\right)\sin\theta d\theta\right]^r \\
&\quad\leqq A_r\int_0^\pi\left(\int_{-\pi}^{\pi} U(\theta,\phi)d\phi\right)^r \sin^r\theta d\theta \\
&\quad\leqq A_r\int_0^\pi\left(\int_{-\pi}^{\pi} U(\theta,\phi)d\phi\right)^r \sin\theta d\theta.
\end{aligned}$$

Next, defining

(11) $$R(\rho, \theta) = 2(1-\rho)^{1+1/r}\theta^{1-1/r}[(1-\rho)^2+\theta^2]^{-3/2}$$

for the interval $(\frac{1}{2}, 1)$, since $\sin\theta \leqq \theta$ and $\sin\frac{1}{2}\theta \geqq \theta/\pi$ over $(0, \pi)$, we have

$$\begin{aligned} Q(\rho,\theta) &\leqq 2(1-\rho)[(1-\rho)^2+2\theta^2/\pi^2]^{-3/2}\sin\theta \\ &= 2^{-1/2}\pi^3(1-\rho)[\tfrac{1}{2}\pi^2(1-\rho)^2+\theta^2]^{-3/2}\sin\theta \\ &\leqq 2^{-1/2}\pi^3(1-\rho)[(1-\rho)^2+\theta^2]^{-3/2}\sin\theta \\ &\leqq 2^{-3/2}\pi^3(1-\rho)^{-1/r}\sin^{1/r}\theta\, R(\rho,\theta), \end{aligned}$$

and so

(12) $$\begin{aligned} &\int_{1/2}^{1}(1-\rho)(u(\rho))^r d\rho \\ &= \int_{1/2}^{1}(1-\rho)\left[\frac{1}{4\pi}\int_0^{\pi}\left(\int_{-\pi}^{\pi}U(\theta,\phi)d\phi\right)Q(\rho,\theta)d\theta\right]^r d\rho \\ &\leqq A_r\int_{1/2}^{1}\left[\frac{1}{4\pi}\int_0^{\pi}R(\rho,\theta)\left(\int_{-\pi}^{\pi}U(\theta,\phi)d\phi\right)\sin^{1/r}\theta\right]^r d\rho. \end{aligned}$$

Defining $f(x)=\sin^{1/r}x\int_{-\pi}^{\pi}U(x,\phi)d\phi$ for $0\leqq x\leqq\pi$, $f(x)=0$ for $x>\pi$, and

$$K(x, y) = (2\pi)^{-1}x^{1-1/r}y^{1+1/r}(x^2+y^2)^{-3/2},$$

the functions $f(x)$, $K(x, y)$ satisfy the conditions of the lemma (with $k=\frac{1}{2}\pi^{-3/2}\Gamma(1-1/r)\Gamma(1/2+1/r)$). On replacing x by θ, y by $1-\rho$, (2) and (11) give

(13) $$\begin{aligned} &\int_{1/2}^{1}\left[\frac{1}{4\pi}\int_0^{\pi}R(\rho,\theta)\left(\int_{-\pi}^{\pi}U(\theta,\phi)d\phi\right)\sin^{1/r}\theta\right]^r d\rho \\ &\leqq A_r\int_0^{\pi}\left(\int_{-\pi}^{\pi}U(\theta,\phi)d\theta\right)^r\sin\theta d\theta. \end{aligned}$$

Combination of (12) and (13) now gives

(14) $$\int_{1/2}^{1}(1-\rho)(u(\rho))^r d\rho \leqq A_r\int_0^{\pi}\left(\int_{-\pi}^{\pi}U(\theta,\phi)d\phi\right)^r \sin\theta d\theta,$$

and addition of (10) and (14) yields the desired inequality (9).

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