NORMAL OPERATORS

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Let W be a hyperbolic Riemann surface with Wiener harmonic boundary Γ. Let $\Gamma=\alpha\cup\beta$ where α and β are disjoint compact sets. Denote by $K(z, p)$ the Wiener harmonic kernel on $W\times\Gamma$ and let μ be the harmonic measure on Γ. If γ denotes a dividing cycle on W homologous to α, then $(\int_\gamma * d_z K(z, p))d\mu(p)$ is a measure on Γ which we shall denote by $d\nu$. According to Nakai and Sario [6], a normal operator T is a linear operator from $C(\alpha)$ into $C(\beta)$ such that

(1.1) $$Tf \geqq 0 \qquad \text{if } f \geqq 0,$$

(1.2) $$T1 = 1,$$

(1.3) $$\int_\beta Tf d\nu = \int_\alpha f d\nu.$$

1. Let S denote the covex set of all normal operators from $C(\alpha)$ into $C(\beta)$. Ahlfors conjectured that a normal operator T is extreme if and only if it is induced by a measure preserving transformation or equivalently carries characteristic functions into characteristic functions [7]. Savage [10] showed this was false even in the simple case of the annulus $1<|z|<2$ with α and β lying over $|z|=1$ and 2, respectively. Nakai and Sario gave an example where the Ahlfors conjecture is valid. In their example, S could be identified with the set of all n by n doubly stochastic matrices. It would be interesting to come up with an example where S could be identified with the set of all infinite doubly stochastic matrices. Since in this case the extreme doubly stochastic matrices, which as is well known, are the permutation matrices, carry characteristic functions into characteristic functions, the Ahlfors conjecture would be valid in this case as well. We shall give such an example.

Take a Riemann surface $R\in O_{HB}^\infty-\bigcup_{n=1}^\infty O_{HB}^n$ (see [4]) and remove a disk Δ, and form the double W of $R-\bar\Delta$ about $\partial\Delta$. Let α be the Wiener harmonic boundary of R and β the symmetric image of α in W. Then disregarding sets of harmonic measure zero as we may, α and β consist of countably many points. Consequently S can be identified with the set of all infinite doubly stochastic matrices.

2. In the case of the annulus $1<|z|<2$ with α and β lying over $|z|=1$ and 2 respectively, we may regard a normal operator as a linear operator on $L^\infty[0, 1]$ which satisfies (1.1), (1.2), and (1.3). As

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is well known, we may regard T as a contraction operator on $L^p[0, 1]$ for $1 \leqq p \leqq \infty$ satisfying (1.1)–(1.3). Révész [9] showed (essentially) that every doubly stochastic matrix is an integral over the set of permutation matrices. We shall generalize this result to the set S of normal operators T regarded as operators on $L^p[0, 1]$, $1 < p \leqq \infty$. For $1 < p < \infty$, we shall endow S with the weak operator topology, and for $p = \infty$, we shall endow S with the weak* operator topology. In the weak operator topology, a basic open neighborhood of $T_0 \in S$ is a set of the form

$$\left\{ T \in S: \left| \int_0^1 f_k T g_k - \int_0^1 f_k T_0 g_k \right| < \epsilon,\ k = 1, \cdots, n \right\},$$

where $f_k \in L^q[0, 1]$, $g_k \in L^p[0, 1]$, and $\epsilon > 0$. In the weak* operator topology, a basic open neighborhood of T_0 is a set of the form

$$\left\{ T \in S: \left| \int_0^1 f_k T g_k - \int_0^1 f_k T_0 g_k \right| < \epsilon,\ k = 1, \cdots, n \right\},$$

where $g_k \in L^\infty[0, 1]$ and $f_k \in L^1[0, 1]$. For $1 < p < \infty$, Brown [3] showed that S is compact and metrizable. For $p = \infty$, the compactness follows from a theorem of Kadison [5].

For $1 < p < \infty$, it follows from Choquet's theorem [8] that for each $T_0 \in S$ there exists a probability measure $\mu(T, T_0)$ on S such that $\mu(T, T_0)\ (S - \operatorname{Ex} S) = 0$, and for each continuous linear functional L on $[L^p[0, 1]\ ]$, $L(T_0) = \int_{\operatorname{Ex} S} L(T) d\mu(T, T_0)$, where $\operatorname{Ex} S$ denotes the set of extreme points of S and $[L^p[0, 1]\ ]$ denotes the space of all bounded linear operators on $L^p[0, 1]$. For $p = \infty$, it follows from the Bishop-de Leeuw theorem [8] that there exists a nonnegative measure $\mu(T, T_0)$ on the σ-ring of subsets of S which is generated by $\operatorname{Ex} S$ and the Baire sets such that $\mu(T, T_0)(S) = \mu(T, T_0)(\operatorname{Ex} S) = 1$ and $L(T_0) = \int_{\operatorname{Ex} S} L(T) d\mu(T, T_0)$ for each continuous linear functional L on $[L^p[0, 1]\ ]$. In either case

$$\int_{\operatorname{Ex} S} L(T) d\mu(T, T_0) = L \int_{\operatorname{Ex} S} T d\mu(T, T_0).$$

Since this holds for every continuous linear functional L, we obtain the following result.

THEOREM. *If* $T_0 \in S$ *and* $1 < p \leqq \infty$, *then*

$$T_0 = \int_{\operatorname{Ex} S} T d\mu(T, T_0) \tag{2.1}$$

where $\mu(T, T_0)(\operatorname{Ex} S) = \mu(T, T_0)(S) = 1$.

3. Let Φ denote the set of all $T \in S$ such that T is induced by an invertible measure preserving transformation. For $1 < p < \infty$, Brown [3] showed that $S = \overline{\Phi}$. We shall prove the same result for $p = \infty$.

For the proof, we observe that since the continuous functions are dense in $L^1[0, 1]$, we may take the f_k's which appear in the definition of the weak* operator topology to be continuous. Let $h_k(x, y) = f_k(x)g_k(y)$ where g_k has the same meaning as in §2. Since we can ignore sets of measure zero, we may assume that g_k is bounded. Hence given $\epsilon > 0$, there exist disjoint measurable sets $X_1, \cdots, X_n$ such that $[0, 1] = \bigcup_{i=1}^n X_i$ and the oscillation of each h_k on $X_i \times X_j$, $i, j = 1, \cdots, n$, is less than ϵ. The remainder of the proof is the same as that given by Brown [3] for the case $1 < p < \infty$.

4. We shall now take $p = 2$. As we remarked in §2, S is compact in the weak operator topology. The question naturally arises as to whether S is compact in the uniform operator topology. We shall prove that this is not the case.

THEOREM. *S is not compact in the uniform operator topology.*

PROOF. Since the strong operator topology is weaker than the weak operator topology, it suffices to prove that S is not compact in the strong operator topology. Let Φ_1 denote the set of all $T \in S$ such that T is induced by a measure preserving transformation (not necessarily invertible).

If S were compact in the strong operator topology, then since S is the closed convex hull of Φ in the strong operator topology (see [3]), it would follow from the converse to the Krein-Milman theorem that the extreme points of S would be contained in $\overline{\Phi}$. But $\overline{\Phi} = \Phi_1$ (see [3]). Hence every extreme point of S would be induced by a measure preserving transformation. Since the Ahlfors conjecture is known to be false for the annulus $1 < |z| < 2$, we have reached a contradiction which proves the theorem.

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