PRODUCT SOLUTIONS FOR SIMPLE GAMES. II

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1. **Introduction.** In this paper we continue our study of the investigation of product solutions for compound simple games. By a compound simple game we mean one that is built up out of two or more component simple games. The concept of compound simple games is apparently due to Shapley. Shapley [2] and the author [1], [4], [5] have obtained product solutions for compound simple games by combining the solutions of the component games. During this process we impose that the subsolutions will have to satisfy a certain semimonotonic ∂-monotonic property. In this paper we obtain a new class of product solutions for the game $H\otimes K$ with $H=V_n\otimes B_1$, where V_n denotes the homogeneous weighted majority game $[1, 1, 1, \cdots, 1, n-2]_h$ consisting of n players, B_1 denotes the 1-person pure bargaining game and K any arbitrary simple game.

2. **Definitions and notations.**

Simple games. We shall denote a simple game by the symbol, $\Gamma(P, W)$, where P is a finite set (players) and W is a collection of subsets of P (the winning coalitions). We demand that $P\in W$ and the empty set is not an element of W.

Let $\Gamma(P_1, W_1)$ and $\Gamma(P_2, W_2)$ be two simple games with $P_1\cap P_2=\emptyset$ and let $P=P_1\cup P_2$. Then the product $\Gamma(P_1, W_1)\otimes\Gamma(P_2, W_2)$ (for simplicity we will write $P_1\otimes P_2$) is defined as the game $\Gamma(P, W)$ where W consists of all $S\subseteq P$ such that $S\cap P_i\in W_i$ for $i=1, 2$. By an imputation we mean a real nonnegative vector x such that $\sum_{i\in P}x_i=1$. A_P will stand for the collection of all imputations. We recall that a solution of the game $\Gamma(P, W)$ is a set X of imputations such that $X=A_P-\operatorname{dom} X$ where dom X denotes the set of all $y\in A_P$ such that for some $x\in X$, the set $\{i\mid x_i>y_i\}$ is an element of W. The notations dom_1 and dom_2 will be used for domination with respect to special classes W_1 and W_2.

Definition. A parameterized family of sets of imputations $Y(\alpha)$: $0\leqq\alpha\leqq 1$ will be called semimonotonic if for every α, β, x such that $0\leqq\alpha\leqq\beta\leqq 1$ and $x\in Y(\beta)$ there exists $y\in Y(\alpha)$ with $\alpha y\leqq\beta x$.

Definition. A semimonotonic family is called ∂-monotonic $(0\leqq\partial\leqq 1)$ if for every α, β, y such that $\partial\leqq\alpha\leqq\beta\leqq 1$ and $y\in Y(\alpha)$ there exists $x\in Y(\beta)$ with $\alpha y\leqq\beta x$.

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We call a 0-monotonic family fully monotonic. [In general, ∂ will stand for any positive number with $0<\partial<1$ unless otherwise stated.]

Let $P=P_1\cup P_2$ and let

$$A_{P_i} = \left\{x: x\in A_P, \sum_{j\in P_i} x_j = 1\right\} \qquad \text{for } i = 1, 2.$$

DEFINITION. Let X be a solution to the product of simple games $P_1\otimes P_2$. Call X a product solution if the following conditions are met.

(i) There exists a semimonotonic family $\{Y_i(\alpha): 0\leqq\alpha\leqq 1\}$ such that $Y_i(\alpha)$ are solutions to P_i for all α except $\alpha=1$ where $i=1, 2$.

(ii) $X=\bigcup_{0\leq\alpha\leq 1} X_1(\alpha)\times_\alpha X_2(1-\alpha)$ where $X_i(\alpha)=A_{P_i}-\text{dom}_i\, Y_i(\alpha)$ and $X_1(\alpha)\times_\alpha X_2(1-\alpha)=\{z: z=\alpha x_1+(1-\alpha)x_2$ for some $x_1\in X_1(\alpha)$, $x_2\in X_2(1-\alpha)\}$.

DEFINITION. Let X be any subset of A_P. Call X an externally stable set if $X\cup\text{dom } X=A_P$. Call X an internally stable set if $X\cap\text{dom } X=\emptyset$. [Here of course we assume $\Gamma(P, W)$ to be a simple game.] Call X a solution if X is both externally stable and internally stable. V_n will always stand for the homogeneous weighted majority game $[1, \cdots, 1, n-2]_h$ consisting of n players. H will stand for any game of the form $V_n\otimes B_1$ where B_1 denotes the 1-person pure bargaining game. K will denote an arbitrary simple game.

3. We now write down the solutions of V_n which are completely known [3, pp. 472–495]. They are classified in three groups.

I. The finite set

$$\left(\frac{1}{n-1}, \frac{1}{n-1}, \cdots, \frac{1}{n-1}, 0\right), \quad \left(\frac{1}{n-1}, 0, 0, 0, 0, \frac{n-2}{n-1}\right),$$

$$\left(0, \frac{1}{n-1}, \cdots, 0, \frac{n-2}{n-1}\right), \cdots, \quad \left(0, 0, 0, 0, \cdots, \frac{1}{n-1}, \frac{n-2}{n-1}\right).$$

II. Let C be any constant with $0\leqq C<1-1/(n-1)$

$$\left\{(x_1, x_2, \cdots, x_{n-1}, C) \mid x_i \geqq 0, \sum x_i = 1 - C\right\}.$$

III. Let S_* be any nonempty proper subset of $\{1, 2, \cdots, n-1\}$. Let $a_1, a_2, \cdots, a_{n-1}$ be nonnegative real numbers which satisfy the following properties:

(i) $\sum_1^{n-1} a_i=1$,

(ii) $a_*=\text{Min}_{1\leq i\leq n-1}\, a_i$, then $a_i=a_*$ for all $i\in S_*$ and $a_i>a_*$ for $i\in\{1, 2, \cdots, n-1\}-S_*$.

As a consequence of (i) and (ii) we have $a_*<1/(n-1)$. Let p be the number of elements in S_* and let $c=1-a_*$, $c^*=1-pa_*$. The following set consisting of (1) and (2) constitutes a solution to V_n.

(1) For $i\in S_*$,

$$\boldsymbol{a}^i = \{\overset{i}{a}_1, \overset{i}{a}_2, \cdots, \overset{i}{a}_n\}$$

where

$$\begin{aligned} \overset{i}{a}_j &= a_i = a_* && \text{for } j = i, \\ &= c && \text{for } j = n, \\ &= 0 && \text{otherwise.} \end{aligned}$$

(2) $\boldsymbol{a}(y)=\{a_1(y), a_2(y), \cdots, a_n(y)\}$ where $0\leqq y\leqq c^*$;

$$\begin{aligned} a_i(y) &= a_i = a_* && \text{for } i \in S_*, \\ &= y && \text{for } i = n, \\ &= a_i(y) && \text{for } i \in \{1, 2, \cdots, n-1\} - S_*, \end{aligned}$$

where $a_i(y)$ for $i\in\{1, 2, \cdots, n-1\}-S_*$ are functions whose domain of definition is $[0, c^*]$ .They also have the following properties

$$a_i(0) = a_i, \qquad a_i(c^*) = 0$$

and

$$|a_i(y_2) - a_i(y_1)| \leqq |y_2 - y_1|.$$

I, II and III exhaust all possible solutions to V_n.

Remark 1. It is not hard to check that no semimonotonic family drawn from this list can include representatives from more than one of the three categories I–III, hence the only possible variation within such a family is in the value of C if the family is from the second group or the variation will be in choosing S_*, the nonnegative real numbers $a_1, a_2, \cdots, a_{n-1}$ and the functions $a_i(y)$ for $i\in\{1, 2, \cdots, n-1\}$ $-S_*$ if the family is from the third group.

Remark 2. Setting $a_*=1/(n-1)$ in III or II produces internally stable sets (not solutions) that are monotonically related to the solutions nearby. Using this fact we will give an example of a solution for compound simple games which is not fully monotonic in the last section. [Recall the fact that a set X is internally stable if $X\cap\text{dom}\, X=\phi$.]

4. We will now state the theorems.

Theorem 1. *Let* $\{X_1(\alpha): 0\leqq\alpha\leqq 1\}$ *be any* ∂*-monotonic family of product solutions to the game* $H=V_n\otimes B_1$ *except that* $X_1(1)$ *need not be externally stable. Then*

$$X = \bigcup_{0 \leq \alpha \leq 1} Z_1(\alpha) \underset{\alpha}{\times} Z_2(1-\alpha)$$

is a solution for $H \otimes K$ *where* K *is any arbitrary simple game and* $Z_1(\alpha) = A_{n+1} - \mathrm{dom}_1 X_1(\alpha)$ *and* $Z_2(1-\alpha) \equiv Z_2$ *is any solution of* K.

THEOREM 2. *Let* $Y_1(\alpha)$ *be* ∂*-monotonic solutions to* V_n. *Then*

$$X = \bigcup_{0 \leq \alpha \leq 1} Z_1(\alpha) \underset{\alpha}{\times} Z_2(1-\alpha)$$

is a solution for $V_n \otimes K$ *where* $Z_1(\alpha) = A_n - \mathrm{dom}_1\ Y_1(1)$ *and* $Z_2(\alpha) \equiv Z_2$ *is any solution of* K.

REMARK 3. External stability of Theorems 1 and 2 can be established as in the case of Theorem 5 of Shapley (see [2, pp. 282–283]) or as in [1] since the proof depends only on the semimonotonic property of $X_i(\alpha)$ and the external stability of $Z_i(\alpha)$.

REMARK 4. Theorem 2 does not say much. This is because every solution that satisfies the conditions of Theorem 2 in fact has the property of full monotonicity even in the 'not required' range. However, using Theorem 1, we can construct solutions which will be ∂-monotonic but not fully monotonic.

PROOF OF THEOREM 1. We will now show that X is internally stable. *We will give the proof when* $H = V_4 \otimes B_1$. *The same proof with some minor modifications applies when* $H = V_n \otimes B_1$ *for general* n.

Case 1. Suppose for infinitely many m, with $\alpha^{(m)} \uparrow 1$, $X_1(\alpha^{(m)})$ is of the form

$$\begin{aligned} X_1(\alpha^m) = {} & \{\beta(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0), 1-\beta\} \\ & \cup \{\beta(\tfrac{1}{3}, 0, 0, \tfrac{2}{3}), 1-\beta\} \\ & \cup \{\beta(0, \tfrac{1}{3}, 0, \tfrac{2}{3}), 1-\beta\} \\ & \cup \{\beta(0, 0, \tfrac{1}{3}, \tfrac{2}{3}), 1-\beta\} \quad \text{where } \beta \text{ runs over from 0 to 1.} \end{aligned}$$

This representation is possible since we are assuming that the $X_1(\alpha)$'s are product solutions. In this case, since the family is ∂-monotonic and since all but a finite number of $\alpha^{(m)}$'s are greater than ∂ it follows that $X_1(1)$ is also of the same form as $X_1(\alpha^{(m)})$'s. In other words $X_1(1)$ is a solution. Hence internal stability follows via Theorem 5 of Shapley [2].

Case 2. Let $\alpha^{(m)} \uparrow 1$ and

$$X_1(\alpha^{(m)}) = \{\beta(x_1, x_2, x_3, C_\beta^{(m)}), 1-\beta \mid 0 \leq \beta < 1,$$
$$x_i \geqq 0, \sum x_i = 1 - C_\beta^{(m)}\} \cup Y_1^{\alpha^m}(1),$$

where $Y_1^{\alpha^m}(1)$ need not be an internally stable set for V_4 and $0 \leqq C_\beta^m < 2/3$. Consider the set N_β where

$$N_\beta = \{x \mid x = \beta(x_1, x_2, x_3, \overset{0}{C_\beta}), 1 - \beta \text{ and there exists a sequence } x_{mK} \in X_1(\alpha^{mK}) \text{ such that } \alpha^{mK} x_{mK} \uparrow x\}.$$

It is easy to check that N_β is nonempty.

It is not hard to check that the closure of $X_1(1)$—written as $\overline{X}_1(1)$—contains the set N_β. This is a consequence of the assumption that the family $X_1(\alpha)$ is ∂-monotonic.

At this point we would like to make another observation, namely $\overline{X}_1(1)$ together with $\{X_1(\alpha): 0 \leqq \alpha < 1\}$ is a semimonotonic family and hence $\overline{X}_1(1)$ is also internally stable.

If $C_\beta^0 < 2/3$ for every β, $\bigcup N_\beta = \overline{X}_1(1)$ and $\overline{X}_1(1)$ is a solution for H, internal stability follows by the theorem of Shapley.

Let $C_\beta = 2/3$ for at least one β .To complete the proof of internal stability in his case it is sufficient to establish that there does not exist any vector $x \in X_1(\alpha)$ with αx dominating y where $y \in Z_1(1) - \overline{X}_1(1)$. Let if possible,

$$\alpha x > y \text{ via } \overline{145} \text{ with } y \in Z_1(1) - \overline{X}_1(1).$$

[$\alpha x > y$ via $\overline{145}$ means, $\alpha x_i > y_i \, \forall i = 1, 4, 5.$]

Let $y = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)$ and

$$X(\alpha) = [\beta(x_1, x_2, x_3, C_\beta), 1 - \beta].$$

[For the sake of simplicity we will not write the possible values of β and x_1, x_2, x_3.]

Choose any β' with $\alpha(1-\beta) > 1-\beta' > \epsilon_5$ where

$$\alpha[\beta(x_1', x_2', x_3', C_\beta), 1 - \beta] > y \text{ via } \overline{145}.$$

Let $N_{\beta'} = [\beta'(x_1, x_2, x_3, C_{\beta'}^0), 1-\beta']$. If $\beta' C_{\beta'}^0 \leqq \epsilon_4$, then there exist $w \in N_{\beta'}$ such that $\alpha x > w$ via $\overline{145}$. This will mean $\alpha x > w \geqq \alpha z$ via $\overline{145}$, contradicting the internal stability of $X_1(\alpha)$.

Hence we will assume $\beta' C_{\beta'}^0 > \epsilon_4$. This means $\beta'(1 - C_{\beta'}^0)$ is less than or equal to ϵ_1, ϵ_2 and ϵ_3, otherwise there will be an element in $N_{\beta'}$ dominating y, thereby contradicting the assumption regarding y. Now a suitable $x^1 \in X(\alpha)$ can be obtained with

$$\alpha x' > \{\beta'(1 - \overset{0}{C_{\beta'}}, 0, 0, \overset{0}{C_{\beta'}}), 1 - \beta'\} \text{ via } \overline{1235}$$

which will once again contradict the internal stability of $X_1(\alpha)$. Similar contradictions can be reached if $\alpha x > y$ via $\overline{245}$ or $\overline{345}$ or $\overline{1235}$.

Thus the proof of internal stability is complete in the case.

Case 3. Let $\alpha^{(m)} \uparrow 1$ and

$$X_1(\alpha^{(m)}) = [\beta(\overset{(m)}{a_\beta}, \overset{(m)}{a_\beta}, \overset{(m)}{a_\beta}(y), y), 1 - \beta] \cup Y_1^{\alpha^{(m)}}(1)$$

where $0 \leqq \beta < 1$, $Y_1^{\alpha^m}(1)$ need not be an internally stable set for V_4 and y runs from 0 to $1-2a_\beta^{(m)}$ for every fixed β. Also note that $0 \leqq a_\beta^{(m)} < 1/3$.

Let (w.l.g.) $a_\beta^m \to a_\beta^0$. Consider now the following set N_β

$$N_\beta = \{x \mid x = \{\beta(\overset{0}{a_\beta}, \overset{0}{a_\beta}, \overset{0}{a_\beta}(y), y), 1 - \beta\}$$
$$\text{and there exists } x^{m_k} \in X_1^{\alpha^{m_k}} \text{ such that } \alpha^{m_k} x^{m_k} \uparrow x\}.$$

For every β, $\{a_\beta^m(y)\}$ is a collection of equicontinuous and uniformly bounded functions defined over the compact set

$$A = \bigcap_{m=1}^{\infty} \overset{m}{A_\beta} \quad \text{where} \quad \overset{m}{A_\beta} = [0, 1 - 2\overset{m}{a_\beta}].$$

It is trivial to check that this intersection is precisely the interval $[0, 1-2a_\beta^0]$. So we can assert without loss of generality $a^m(y) \to a_\beta^0(y)$ uniformly over A. If for every β, $a_\beta^0 < 1/3$, we are through. So we will assume $Z_1(1) - \overline{X}_1(1) \neq \varnothing$. We will now prove that there exists no vector $x \in X(\alpha)$ with αx dominating y where $y \in Z_1(1) - \overline{X}_1(1)$. Let if possible

$$\alpha x > y = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5) \text{ via say } \overline{1235}$$

i.e.

$$\alpha[\beta(x_1, x_2, x_3, x_4), 1 - \beta] > y.$$

Let $X_1(\alpha)$ be of the form

$$\{\beta(a_\beta, a_\beta, a_\beta(t), t), 1 - \beta\}.$$

Choose and fix one β' such that $\alpha(1-\beta) > 1-\beta' > \epsilon_5$

$$N_{\beta'} = \{\beta'(\overset{0}{a_{\beta'}}, \overset{0}{a_{\beta'}}, \overset{0}{a_{\beta'}}(t), t), 1 - \beta'\}.$$

If $\beta' a_{\beta'}^0 \leqq \epsilon_1$ and ϵ_2 then internal stability of $X_1(\alpha)$ will be contradicted. Hence, we will assume $\beta' a_{\beta'}^0 > \epsilon_1$. This will mean $\beta'(1-2a_{\beta'}^0) \leqq \epsilon_4$ otherwise there will be an element $w \in N_{\beta'}$ such that w will dominate y thereby contradicting the assumption regarding y. If $\beta' a_{\beta'}^0 > \epsilon_2$ and $\beta'(1-2a_{\beta'}^0) > \epsilon_3$, then once again there will be a contradiction regarding the assumption that $y \in Z_1(1) - \overline{X}_1(1)$.

If $\beta' a_{\beta'}^0 \leqq \epsilon_2$ then we can find $u, u' \in X_1(\alpha)$, $w \in N_{\beta'}$ such that $\alpha u > w \geqq \alpha u'$ via 245 i.e. $u > u'$ via 245 and this leads to a contradiction.

If $\beta'(1-2a_{\beta'}^0) \leqq \epsilon_3$, we can find $u, u' \in X(\alpha)$ with $u > u'$ via $\overline{345}$ and hence a contradiction.

If $X_1(\alpha)$ is of the form

$$\{\beta(a_\beta, \overset{1}{a}_\beta(t), \overset{2}{a}_\beta(t), t), 1 - \beta\},$$

then also one can show the impossibility of αx dominating y with $y \in Z_1(1) - \overline{X}_1(1)$. Similar contradictions can be reached if $N_{\beta'}$ is of the form

$$N_{\beta'} = \{\beta'(\overset{0}{a}_{\beta'}, \overset{1}{a}_{\beta'}(t), \overset{2}{a}_{\beta'}(t), t), 1 - \beta'\}.$$

Thus the proof of internal stability of X is complete.

REMARK 5. During the course of the proof we have omitted certain minor details. For example if $X_1(\alpha') = \{\beta(x_1,x_2,x_3,c_\beta), 1-\beta\}$ then for all α with $(2\alpha'/3) < \alpha \leqq \alpha'$, $X(\alpha)$ will also be of the same form as $X(\alpha')$. This is a consequence of the fact that the $X(\alpha)$'s form a semimonotonic family and are product solutions.

REMARK 6. Theorem 1 includes Theorem 3.2 in [1] where we have obtained product solutions for the game $H \otimes K$, $H = V_3 \otimes B_1$ and K is any simple game.

5. The following example shows that solutions to product simple games can be found which need not have the property of full monotonicity. Consider the game $H = V_4 \otimes B_1$ and define for $0 \leqq \alpha \leqq 3/4$,

$$X(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta)$$

where

$$Y(\beta) = \{(\beta x_1, \beta x_2, \beta x_3, 2\beta^{3/2}/3, 1 - \beta) \mid x_i \geqq 0, \textstyle\sum x_i = 1 - 2\beta^{1/2}/3\} \quad \text{for } 0 \leqq \beta < 1,$$

$$Y(1) = \{(x_1, x_2, x_3, \tfrac{2}{3}, 0) \mid x_i \geqq 0, \textstyle\sum x_i = \tfrac{1}{3}\} \cup \{\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0\}.$$

For $3/4 < \alpha \leqq \alpha_0$, where α_0 is so chosen that

$$\alpha_0\left(1 - \frac{7}{6}\,\frac{1}{1+\alpha_0}\right) = \frac{1}{3},$$

define

$$X(\alpha) = \bigcup_{0 \leq \beta \leq 1} Y(\beta),$$

$$Y(\beta) = \left\{\left(\beta x_1, \beta x_2, \beta x_3, \frac{7}{6}\,\frac{\beta^{3/2}}{1+\alpha}, 1 - \beta\right) \middle| \, x_i \geqq 0, \right.$$

$$\left. \sum x_i = 1 - \frac{7}{6}\,\frac{\beta^{1/2}}{1+\alpha}\right\}.$$

For $\alpha_0 < \alpha \leqq 1$ define

$$X(\alpha) = \bigcup_{0 \leq \beta \leq 1} \left\{ \left(\beta x_1, \beta x_2, \beta x_3, \left(1 - \frac{1}{3\alpha}\right)\beta^{3/2}, 1 - \beta \right) \mid x_i \geqq 0, \right.$$

$$\left. \sum x_i = 1 - \left(1 - \frac{1}{3\alpha}\right)\beta^{1/2} \right\}.$$

Now it is not hard to check that the family $X(\alpha)$ is semimonotonic and that each $X(\alpha)$ is a product solution to the game H except $X(1)$. $X(1)$ is not externally stable because,

$$(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0) \notin X(1) \cup \operatorname{dom} X(1).$$

If $X^1(1) = X(1) \cup (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$, then $X^1(1)$ is a solution to H. But $X^1(1)$ together with $\{X(\alpha): 0 \leqq \alpha < 1\}$ is not semimonotonic, for, corresponding to $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0) \in X^1(1)$ there exists no element $x \in X(\alpha)$ for any $\alpha > 3/4$ with $\alpha x \leqq (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$.

Further the family $X(\alpha): 0 \leqq \alpha \leqq 1$ is not fully monotonic because corresponding to the element $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0) \in X(3/4)$ there exists no element $y \in X(1)$ with the property that $y \geqq (3/4)(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$. It is not hard to check that conditions of Theorem 1 are satisfied for this family $X(\alpha)$. Hence this family can be used to produce product solutions for the game $H \otimes K$. This is the example promised at the end of §3.

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