CHARLIER SPACES OF ENTIRE FUNCTIONS

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The paper is concerned with examples of Hilbert spaces whose elements are entire functions and which have these properties:

(H1) Whenever $F(z)$ is in the space and has a nonreal zero w, the function $F(z)(z-\bar{w})/(z-w)$ belongs to the space and has the same norm as $F(z)$.

(H2) For each nonreal number w, the linear functional defined on the space by $F(z) \to F(w)$ is continuous.

(H3) The function $F^*(z) = \bar{F}(\bar{z})$ belongs to the space whenever $F(z)$ belongs to the space, and it always has the same norm as $F(z)$.

The theory of these spaces is related to the theory of entire functions $E(z)$ which satisfy the inequality

$$|E(x-iy)| < |E(x+iy)|$$

for $y > 0$. If $E(z)$ is such a function, we write $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions which are real for real z, and

$$K(w, z) = [B(z)\bar{A}(w) - A(z)\bar{B}(w)]/[\pi(z-\bar{w})].$$

Let $\mathcal{H}(E)$ be the set of entire functions $F(z)$ such that

$$\|F\|^2 = \int_{-\infty}^{+\infty} |F(t)/E(t)|^2 dt < \infty$$

and such that

$$|F(z)|^2 \leq \|F\|^2 K(z, z)$$

for all complex z. Then $\mathcal{H}(E)$ is a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). For each complex number w, $K(w, z)$ belongs to $\mathcal{H}(E)$ as a function of z, and the identity

$$F(w) = \langle F(t), K(w, t)\rangle$$

holds for all elements $F(z)$ of $\mathcal{H}(E)$. A Hilbert space, whose elements are entire functions, which satisfies the axioms (H1), (H2), and (H3), and which contains a nonzero element is equal isometrically to a space $\mathcal{H}(E)$.

The spaces now studied are finite dimensional spaces related to Charlier's orthogonal polynomials. They are characterized by an

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identity involving a positive parameter a. The identity implies a recurrence relation for the defining functions $A(z)$ and $B(z)$ of the space.

THEOREM 1. *Let $\mathcal{H}(E)$ be a given space such that $E(z)$ has a nonzero value at the origin, and let a be a given positive number. Assume that the functions $z[F(z)-F(z-1)]$ and $F(z+1)-F(z)$ belong to the space whenever $F(z)$ belongs to the space, and that the identity*

$$\begin{aligned}&\langle t[F(t) - F(t-1)] - a^2[F(t+1) - F(t)], G(t)\rangle \\ &\qquad = \langle F(t), t[G(t) - G(t-1)] - a^2[G(t+1) - G(t)]\rangle\end{aligned}$$

holds for all elements $F(z)$ and $G(z)$ of the space. Then there exist real numbers u_+, v_+, u_-, v_- such that the functions

$$S_+(z) = A(z)u_+ + B(z)v_+ \text{ and } S_-(z) = A(z)u_- + B(z)v_-$$

are linearly independent and satisfy the recurrence relations

$$\begin{aligned}\lambda_+ S_+(z) &= z[S_+(z) - S_+(z-1)] - a^2[S_+(z+1) - S_+(z)],\\ \lambda_- S_-(z) &= z[S_-(z) - S_-(z-1)] - a^2[S_-(z+1) - S_-(z)],\\ \lambda_+ S_-(z) &= aS_+(z+1) - aS_+(z),\\ S_+(z) &= a^{-1}zS_-(z-1) - aS_-(z),\end{aligned}$$

for some real numbers λ_+ and λ_- such that $\lambda_+ = 1 + \lambda_-$.

Kummer's confluent hypergeometric function

$$F(a; c; z) = 1 + \frac{a}{1!c} z + \frac{a(a+1)}{2!c(c+1)} z^2 + \cdots$$

is used to construct spaces satisfying the hypotheses of Theorem 1.

THEOREM 2. *If $a > 0$ is given, then the polynomials $\Phi_n(z)$ defined by*

$$\Phi_n(z) = (-a)^{-n}[\Gamma(n-z)/\Gamma(-z)]F(-n; 1 + z - n; a^2)$$

are real for real z and satisfy the identities

$$\begin{aligned}n\Phi_n(z) &= z[\Phi_n(z) - \Phi_n(z-1)] - a^2[\Phi_n(z+1) - \Phi_n(z)],\\ n\Phi_{n-1}(z) &= a\Phi_n(z+1) - a\Phi_n(z),\\ \Phi_{n+1}(z) &= a^{-1}z\Phi_n(z-1) - a\Phi_n(z),\\ z\Phi_n(z) &= an\Phi_{n-1}(z) + (n + a^2)\Phi_n(z) + a\Phi_{n+1}(z).\end{aligned}$$

There exist spaces $\mathcal{H}(E_n)$, $n = 1, 2, 3, \cdots$, satisfying the hypotheses of Theorem 1, such that $\mathcal{H}(E_n)$ is contained isometrically in $\mathcal{H}(E_{n+1})$

for every n, *such that* $\Phi_0(z)$ *spans* $\mathcal{H}(E_1)$, *and such that* $\Phi_n(z)$ *spans the orthogonal complement of* $\mathcal{H}(E_n)$ *in* $\mathcal{H}(E_{n+1})$ *for every* $n>0$. *The spaces can be chosen so that* $\|\Phi_n(t)\|^2=\Gamma(1+n)$ *for every* n. *The identity*

$$e^{a^2}\|F(t)\|^2 = \sum_{n=0}^{\infty} |F(n)|^2 a^{2n}/\Gamma(1+n)$$

then holds for every polynomial $F(z)$.

These are essentially all the spaces which satisfy the hypotheses of Theorem 1.

THEOREM 3. *If* $\mathcal{H}(E)$ *is a space which satisfies the hypotheses of Theorem* 1, *then there exists an entire function* $S(z)$ *which is real for real* z, *has only real zeros, and is periodic of period one, and there exists an index* r *in Theorem* 2 *such that the transformation* $F(z)\to S(z)F(z)$ *is an isometry of* $\mathcal{H}(E_r)$ *onto* $\mathcal{H}(E)$.

These spaces, like those of previous work [2], are related to generalized spaces of square summable power series. Let a and c be numbers such that the coefficients of Kummer's series $F(a; c; z)$ are all positive. By $\mathcal{C}(a; c; z)$ we mean the Hilbert space of power series $f(z)=\sum a_n z^n$ with complex coefficients such that

$$\|f(z)\|^2 = |a_0|^2 + \frac{1!c}{a}|a_1|^2 + \frac{2!c(c+1)}{a(a+1)}|a_2|^2 + \cdots < \infty.$$

The series which belong to $\mathcal{C}(a; c; z)$ converge in the complex plane and represent entire functions. The series $F(a; c; \bar{w}z)$ belongs to the space for all complex numbers w, and the identity

$$f(w) = \langle f(z), F(a; c; \bar{w}z)\rangle$$

holds for every element $f(z)$ of the space.

THEOREM 4. *In Theorem* 2 *if* $f(z)=\sum a_n z^n$ *is a polynomial of degree less than* r, *then its eigentransform* $F(z)=\sum a_n\Phi_n(z)$ *belongs to* $\mathcal{H}(E_r)$ *and*

$$\int_{-\infty}^{+\infty} |F(t)/E_r(t)|^2 dt = \|f(z)\|^2$$

where the norm of $f(z)$ *is taken in* $\mathcal{C}(1; 1; z)$. *Every element of* $\mathcal{H}(E_r)$ *is of this form. The identity*

$$\Gamma(z)F(-z) = \int_0^{\infty} f(-a - t/a)e^{-t}t^{z-1}dt$$

holds for $x>0$ whenever $f(z)$ is a polynomial and $F(z)$ is its eigentransform. Let $f(z)$ and $g(z)$ be polynomials, and let $F(z)$ and $G(z)$ be their eigentransforms. The condition

$$G(z) = z[F(z) - F(z-1)] - a^2[F(z+1) - F(z)]$$

is necessary and sufficient that $g(z)=zf'(z)$. The condition $G(z) = aF(z+1)-aF(z)$ is necessary and sufficient that $g(z)=f'(z)$. The condition $G(z)=a^{-1}zF(z-1)-aF(z)$ is necessary and sufficient that $g(z)=zf(z)$. The condition $G(z)=zF(z)$ is necessary and sufficient that $g(z)=(z+a)f'(z)+a(z+a)f(z)$.

Proof of Theorem 1. Let L_+, L_-, and D be the transformations on entire functions defined by D: $F(z)\to G(z)$ if

$$G(z) = z[F(z) - F(z-1)] - a^2[F(z+1) - F(z)],$$

L_-: $F(z)\to G(z)$ if

$$G(z) = aF(z+1) - aF(z),$$

and L_+: $F(z)\to G(z)$ if

$$G(z) = a^{-1}zF(z-1) - aF(z).$$

A straightforward calculation will show that the commutator identities

$$DL_- - L_-D = - L_-, \quad DL_+ - L_+D = L_+, \quad L_-L_+ - L_+L_- = 1,$$

are satisfied. By hypothesis the restriction of D to the space is a selfadjoint transformation in the space. Since D is everywhere defined in the space, it is bounded. The hypotheses also imply that L_- takes the space into itself. Since the restriction of L_- to the space has a closed graph, it is bounded. If $F(z)$ belongs to the domain of multiplication by z and if D: $F(z)\to G(z)$, then

$$D\colon zF(z) \to zG(z) + zF(z-1) - a^2F(z+1).$$

It follows that the identity

$$\langle tF(t-1) - a^2F(t+1), G(t)\rangle = - \langle F(t), tG(t-1) - a^2G(t+1)\rangle$$

holds whenever $F(z)$ and $G(z)$ belong to the domain of multiplication by z in the space. Since D is selfadjoint, the identity

$$\langle tF(t-1), G(t)\rangle = a^2\langle F(t), G(t+1)\rangle$$

holds whenever $F(z)$ and $G(z)$ belong to the domain of multiplication by z in the space. As in the proof of Theorem 1 of [2], this implies that L_+ acts as a bounded transformation on the domain of multiplication by z. Since the action of $D+aL_+ +aL_-$ coincides with multiplication by z, multiplication by z is a bounded transformation in the space. An argument in the proof of Theorem 1 of [2] will show that the space is finite dimensional.

Let r be the dimension of the space. Since we assume that $E(z)$ has a nonzero value at the origin, there exists an element of the space which has a nonzero value at the origin. Since the transformation $F(z) \to z[F(z)-F(z-1)]$ does not take the space onto itself, it has a nonzero kernel. It follows that there exists a nonzero element $S(z)$ of the space which is periodic of period one. Since $z[F(z+1)-F(z)]$ belongs to the space whenever $F(z)$ belongs to the space, and since $E(z)$ has a nonzero value at the origin, the functions $F(z+1)-F(z)$ and $F(z+1)$ belong to the space whenever $F(z)$ belongs to the space. Since the space is finite dimensional, there exists no zero w of $S(z)$ such that $S(z)/(z+n-w)$ belongs to the space for every $n=0, 1, 2, \cdots$. It follows that there exists no zero w of $S(z)$ such that $S(z)/(z-w)$ belongs to the space. By Problem 88 of [1], $S(z)$ and $S^*(z)$ are linearly dependent, and the elements of the space are the entire functions $F(z)$ such that $F(z)/S(z)$ is a polynomial of degree less than r. We assume that $S(z)$ is chosen of norm one and real for real z.

Let $S_0(z), S_1(z), S_2(z), \cdots$ be the entire functions defined inductively by $S_0(z)=S(z)$ and

$$L_+ \colon S_n(z) \to S_{n+1}(z).$$

The commutator identities imply that

$$D\colon S_n(z) \to nS_n(z)$$

for every n and that

$$L_-\colon S_n(z) \to nS_{n-1}(z)$$

for every $n>0$. It follows that the identity

$$zS_n(z) = anS_{n-1}(z) + (n + a^2)S_n(z) + aS_{n+1}(z)$$

holds for $n>0$, and for $n=0$ with the term in $S_{n-1}(z)$ omitted. It is clear that $S_n(z)/S(z)$ is a polynomial of degree n. So $S_n(z)$ belongs to $\mathcal{H}(E_r)$ when $n<r$. The functions $S_0(z), \cdots, S_{r-1}(z)$ are orthogonal in $\mathcal{H}(E_r)$ since they are eigenfunctions of a selfadjoint operator for distinct eigenvalues. Since multiplication by z is a symmetric transformation, the identity

$$\langle tS_{n-1}(t), S_n(t)\rangle = \langle S_{n-1}(t), tS_n(t)\rangle$$

holds when $0<n<r$. It follows that $\|S_n(t)\|^2=n\|S_{n-1}(t)\|^2$. Since we assume that $\|S_0(t)\|=1$, we can conclude that $\|S_n(t)\|^2=\Gamma(1+n)$.

As in the proof of Theorem 1 of [2], there exists a space $\mathcal{H}(E_{r+1})$, satisfying the hypotheses of Theorem 1, which contains $\mathcal{H}(E_r)$ isometrically, such that $S_r(z)$ spans the orthogonal complement of $\mathcal{H}(E_r)$ in $\mathcal{H}(E_{r+1})$. The theorem now follows from Theorem 23 and Problem 87 of [1].

Proof of Theorem 2. It is clear from the definition of Kummer's series that $\Phi_n(z)$ is a polynomial of degree n which is real for real z. The stated identities for $\Phi_n(z)$ follow from the well-known relations between contiguous hypergeometric series, Erdélyi [3]. Consider the unique inner product on polynomials with respect to which the functions $\Phi_n(z)$ are an orthogonal set and $\|\Phi_n(t)\|^2=\Gamma(1+n)$ for every n. Define L_+, L_-, and D as in the proof of Theorem 1. It is easily verified that the identities

$$\langle DF, G\rangle = \langle F, DG\rangle,$$
$$\langle L_+F, G\rangle = \langle F, L_-G\rangle,$$
$$\langle tF(t), G(t)\rangle = \langle F(t), tG(t)\rangle$$

hold for all polynomials $F(z)$ and $G(z)$. For every $r=1, 2, 3, \cdots$, the polynomials of degree less than r are a Hilbert space of entire functions which satisfies the axioms (H1), (H2), and (H3). By Theorem 23 of [1], the space is equal isometrically to a space $\mathcal{H}(E_r)$. The function $E_r(z)$ has a nonzero value at the origin since the space contains a constant function which has a nonzero value at the origin. The polynomials $z[F(z)-F(z-1)]$ and $F(z+1)-F(z)$ belong to the space whenever $F(z)$ belongs to the space since their degrees do not exceed the degree of $F(z)$. The restriction of D to the space is selfadjoint since the space admits an orthogonal basis of eigenfunctions of D corresponding to real eigenvalues. From this we see that $\mathcal{H}(E_r)$ satisfies the hypotheses of Theorem 1 for every r.

To complete the proof of the theorem, we consider a new inner product on polynomials defined by

$$\langle F(t), G(t)\rangle_1 = \sum_{n=0}^{\infty} F(n)\bar{G}(n)a^{2n}/\Gamma(1+n).$$

It is easily verified that the identity

$$\langle tF(t-1), G(t)\rangle_1 = a^2\langle F(t), G(t+1)\rangle_1$$

holds for all polynomials $F(z)$ and $G(z)$. It follows that the identities

$$\langle DF, G\rangle_1 = \langle F, DG\rangle_1, \qquad \langle L_+F, G\rangle_1 = \langle F, L_-G\rangle_1,$$

hold for all polynomials $F(z)$ and $G(z)$. The proof of Theorem 1 will show that

$$\langle F(t), G(t)\rangle_1 = \kappa\langle F(t), G(t)\rangle$$

for all polynomials $F(z)$ and $G(z)$, where κ is a constant. When $F(z) = G(z) = 1$, we obtain

$$\kappa = \sum_{n=1}^{\infty} a^{2n}/\Gamma(1+n) = e^{a^2}.$$

Proof of Theorem 3. This more general result follows from the above proofs of Theorems 1 and 2.

Proof of Theorem 4. The theorem follows by a routine calculation once it is known that the formula

$$\Gamma(z)\Phi_n(-z) = \int_0^\infty (-a - t/a)^n e^{-t} t^{z-1} dt$$

holds for every n when $x > 0$. The formula is true when $n = 0$ by the definition of the gamma function. A straightforward calculation will show that the functions defined by this integral formula satisfy the recurrence relations of Theorem 2. These functions must therefore coincide with the functions of Theorem 2.

References

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