A NOTE ON COMMUTATIVE SEMIGROUPS[1]

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From Theorem 19.1 of [1] it follows that the lattice of varieties (equational classes) of Abelian groups is distributive. We show that the lattice of varieties of commutative semigroups is not modular by considering an example from the dual lattice of closed commutative semigroup laws. We will assume and use without comment that any given set of laws contains the commutative and associative laws. The binary operation is denoted by juxtaposition. By (a, b), where a and b are positive integers, we will mean the term $x^a y^b$.

Let the sets A', B', and C' of laws be given by the following table.

A'	B'	C'
$(1,9) = (2,8)$	$(2,8) = (3,7)$	$(1,9) = (2,8)$
$(3,7) = (4,6)$	$(4,6) = (5,5)$	$(3,7) = (4,6)$
		$(1,9) = (5,5)$

Let A, B, and C be the closures of A', B', and C' respectively. We denote the closure of $A \cup B$ by $A + B$. Clearly $A \subseteq C$; however, we will show that $(A+B) \cap C \neq A + (B \cap C)$. It is easily seen, using transitivity, that $(1, 9) = (5, 5)$ is in $(A+B) \cap C$.

By considering the reduced free (commutative) semigroup of rank two of the variety defined by B one can find the set B_2 of all laws in x and y that are in B. One can similarly find C_2 and then observe that $B_2 \cap C_2 \subseteq A$. This means that $B_2 \cap C_2$ and hence $B \cap C$ holds in the reduced free semigroup of rank two of the variety defined by A. Hence $A + (B \cap C)$ holds in this semigroup. However, by determining the semigroup, one finds that $(1, 9) = (5, 5)$ does not hold in it. Hence $(1, 9) = (5, 5)$ is not in $A + (B \cap C)$.

We indicate all the nontrivial elements of A_2, B_2, and C_2.

A_2

$$(1,9) = (2,8), (9,1) = (8,2), (3,7) = (4,6), (7,3) = (6,4)$$

$$(a, b) = (c, d) \quad \text{for } a + b, c + d > 10, x^a = x^b, y^a = y^b \quad \text{for } a, b > 10$$

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[1] The example in this paper is contained in the author's Ph.D. thesis written at the University of Nebraska under the direction of H. H. Schneider.

$$B_2$$

$$(2,8) = (3,7), (8,2) + (7,3), (4,6) = (5,5) = (6,4)$$

$$(2,9) = (3,8) = (4,7) = (5,6) = (6,5) = (7,4) = (8,3) = (9,2)$$

$$(a,b) = (c,d) \quad \text{for } a,b,c,d > 1 \quad \text{and} \quad a+b, c+d > 11$$

$$(a,1) = (b,1), (1,a) = (1,b), x^a = x^b, y^a = y^b \quad \text{for } a,b > 11$$

$$C_2$$

$$(1,9) = (2,8) = (5,5) = (8,2) = (9,1), (3,7) = (4,6), (6,4) = (7,3)$$

$$(a,b) = (c,d) \quad \text{for } a+b, c+d > 10, x^a = x^b, y^a = y^b \quad \text{for } a,b > 10$$

A labeling of "equal points" with the same letter on graph paper was helpful. A slightly simpler example showing only nondistributivity can be given. The question of the modularity of variety lattices was suggested by H. Ribeiro.

Reference

1. B. H. Neumann, *Identical relations in groups*. I, Math. Ann. **114** (1937), 506–525.

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