

CHARACTERIZATIONS FOR APPROXIMATELY NORMAL ALGEBRAS

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1. Introduction. Let $\bar{C}(X)$ be the algebra of complex valued continuous functions on a compact Hausdorff space X . A function algebra A is a uniformly closed separating subalgebra of $C(X)$ containing the constants.

DEFINITION 1.1. A is *approximately normal* on X if for any two disjoint closed sets E, F in X and $\epsilon > 0$ there is a function f in A such that $|f| < \epsilon$ on E and $|1 - f| < \epsilon$ on F .

In this note we characterize approximately normal algebras and their essential sets. As applications we obtain a "belonging" result of Wilken [9] and show that the only algebra which restricts to all continuous functions on each of a countable number of closed sets whose union is X is the algebra $C(X)$.

NOTATION. For a closed set $F \subset X$ and $f \in C(X)$, $\|f\|_F$ will denote the sup norm of $f|_F$, the restriction of f to F . A_F will be the uniform closure of the algebra of restrictions, $A|_F = \{f|_F : f \in A\}$. $M(A)$ will denote the maximal ideal space of A and \hat{A} the algebra of Gelfand transforms of A . $A\text{-hull}(F) = \{p \in M(A) : |\hat{f}(p)| \leq \|f\|_F \text{ for all } f \in A\}$. For $p \in M(A)$, a positive regular Borel measure μ_p on X represents p if $\hat{f}(p) = \int f d\mu_p$ for all $f \in A$. For a bounded regular Borel measure μ on X , $\text{supp } \mu$, the closed support of μ , is the complement of the largest open set of $|\mu|$ -measure 0.

2. Characterization of approximate normality. For a closed set F in X , $M(A_F) = A\text{-hull}(F)$ [6]. The following is immediate.

LEMMA 2.1. *Let A be a function algebra on X . Then for any closed set F in X , $A\text{-hull}(F) = \{p \in M(A) : \exists \text{ a representing measure } \mu_p \ni : \text{supp } \mu_p \subset F\}$.*

The following generalizes a result Ryff obtained for the disk algebra using Mergelyan's theorem [7].

LEMMA 2.2. *Let μ be either a representing measure or an extreme point of the ball of A^\perp , the annihilator of the function algebra A . Then if $S = \text{supp } \mu$, $A\text{-hull}(S)$ is connected.*

Received by the editors October 20, 1967.

¹ Part of the material here appears in the author's doctoral dissertation at North-Western University. The research was supported in part by the National Science Foundation. The author wishes to thank Professor G. M. Leibowitz for his guidance throughout this research.

PROOF. It is well known that in either case $S = \text{supp } \mu$ is a set of antisymmetry for A_S [3]. Thus A_S is antisymmetric and it follows from Šilov's idempotent theorem [5] that $M(A_S) = A\text{-hull}(S)$ must be connected.

THEOREM 2.3. *The following are equivalent.*

- (i) A is approximately normal on X .
- (ii) Every representing measure for A on X has connected closed support.
- (iii) For each pair of disjoint closed sets E, F in X , $A\text{-hull}(E \cup F) = A\text{-hull}(E) \cup A\text{-hull}(F)$ and $A\text{-hull}(E) \cap A\text{-hull}(F) = \emptyset$.
- (iv) For each closed set F in X , the canonical map $G' \rightarrow G' \cap F$ from the open closed sets of $M(A_F)$ to the open closed sets of F is 1-1 and onto.
- (v) For each closed set F in X , the canonical map from the components of F to the components of $M(A_F)$ is 1-1 and onto.

If A is approximately normal then the inverse map for (iv) and the map in (v) are both given by taking the A -hull.

PROOF. (i) \Rightarrow (ii). Let μ be a representing measure for A concentrated on X . If $\text{supp } \mu = E \cup F$ where E and F are disjoint closed sets then by approximate normality, the characteristic function of E , χ_E is in $A_{E \cup F}$, contradicting the antisymmetry of $\text{supp } \mu$.

(ii) \Rightarrow (iii). If E, F are disjoint closed sets in X , then by Lemma 2.1 $A\text{-hull}(E \cup F) \subset A\text{-hull}(E) \cup A\text{-hull}(F)$ and so we have equality. If $p \in A\text{-hull}(E) \cap A\text{-hull}(F)$ then there are representing measures μ_1, μ_2 for p concentrated on E, F respectively so that $1/2\mu_1 + 1/2\mu_2$ represents p and does not have connected support, contradicting (ii).

(iii) \Rightarrow (i). If E, F are disjoint closed sets in X then $M(A_{E \cup F}) = A\text{-hull}(E \cup F) = A\text{-hull}(E) \cup A\text{-hull}(F)$ (disjoint). Thus by the Šilov idempotent theorem $\chi_E \in A_{E \cup F}$ and approximate normality follows.

(ii) \Rightarrow (iv). Let F be a closed set in X . There is a 1-1 correspondence between the open closed sets of $M(A_F)$ and the continuous characteristic functions in \hat{A}_F . Since F is the Šilov boundary of A_F this gives a 1-1 map from the open closed sets of $M(A_F)$ into the open closed sets of F which is "intersection." By approximate normality any open closed set in F gives rise to a characteristic function in A_F so that this map is onto.

If G' is open closed in $M(A_F)$ and μ_p represents $p \in G'$ then by antisymmetry $\text{supp } \mu_p \subset G'$. Thus $G' = A\text{-hull}(G' \cap F)$ by Lemma 2.1.

(iv) \Rightarrow (v). We use the fact that the components are intersects of open closed sets. Thus, normality and compactness shows that the map components (F) \rightarrow components ($M(A_F)$) is onto. Similarly assuming (iv), two distinct components in F , C_1 and C_2 are contained in

disjoint open closed sets in F . The corresponding open closed sets in $M(A_F)$ must be disjoint and hence C_1 and C_2 are contained in distinct components in $M(A_F)$, so that the map is 1-1.

(v) \Rightarrow (ii) is clear by Lemma 2.2.

If C' is the component in $M(A_F)$ corresponding to C then C' is an intersection of open closed sets and hence as above if, $p \in C'$ and μ_p represents p , then $\text{supp } \mu_p$ is contained in C' . Thus $C' = A\text{-hull}(C' \cap F)$. Note that $C = C' \cap F$ since by (iv) $C' \cap F$ must be connected.

One notes that (a) $R(X)$ (for a compact set X in the plane) is approximately normal iff X has no interior [9], (b) a pervasive algebra on X is approximately normal iff X is connected, and (c) every Dirichlet algebra is approximately normal. Thus the following result of Coburn [2] originally obtained via Toeplitz operators, follows.

COROLLARY 2.4. *If A is Dirichlet on X then every representing measure for A on X has connected closed support.*

COROLLARY 2.5. *If A is approximately normal then every extreme point of $b(A^\perp)$ has connected closed support.*

The above allows the following conceptual proof of a "belonging" result of Wilken [9].

COROLLARY 2.6. *If A is approximately normal on X and $\{C_\alpha\}$ is the family of components of X and if $f \in C(X)$ has $f|_{C_\alpha} \in A|_{C_\alpha}$ for all α , then $f \in A$.*

PROOF. It suffices to show that f is annihilated by each extreme point of the ball of A^\perp . But each such extreme point is concentrated on a component and thus the result follows from the Krein-Milman theorem.

REMARK. Steen [8] has constructed an essential approximately normal Swiss cheese algebra $R(X)$ on a connected planar set X which is not antisymmetric, so it follows that components are not necessarily antisymmetric for essential approximately normal algebras. Conversely, Corollary 2.5 shows that approximately normal algebras do contain many connected sets of antisymmetry. Although we have not been able to construct a counterexample, it seems unlikely that all closed sets of antisymmetry will be connected.

3. Characterization of the essential set. The *essential set* for an algebra A on X is the smallest closed set supporting every bounded regular Borel measure on X annihilating A . Alternately it is the smallest closed set E_X in X such that $f \in C(X)$ and $f|_{E_X} \in A|_{E_X} \Rightarrow f \in A$ [2].

For A having metrizable ideal space $M = M(A)$, Mullins [4] has characterized the essential set on $M(A)$ by $E_M = M \sim P$ where $P = \{x \in M: \exists \text{ a closed nbhd } \bar{V}_x \text{ of } x \text{ with } A|_{\bar{V}_x} = C(\bar{V}_x)\}$. Such a characterization fails in general for $X \not\subseteq M(A)$, as he points out. However it does hold if A is approximately normal on X . To do this we use the following "2-local" belonging property of approximately normal algebras [9].

LEMMA 3.1. *Let A be approximately normal on X and let $f \in C(X)$. If there are open sets U_1, U_2 such that $X = U_1 \cup U_2$ and $f|_{U_1} \in A|_{U_1}$ and $f|_{U_2} \in A|_{U_2}$ then $f \in A$.*

PROPOSITION 3.2. *If A is approximately normal on X , then $E_X = X \sim P$ where $P = \{x \in X: \exists \text{ a closed nbhd } \bar{V}_x \text{ of } x \text{ with } A|_{\bar{V}_x} = C(\bar{V}_x)\}$.*

PROOF. Evidently $X \sim P \subseteq E_X$. Now suppose $x \in P$. Then choose open sets U and W such that $x \in U \subset \bar{U} \subset W$ and $A|_{\bar{W}} = C(\bar{W})$. Suppose that $f \in C(X)$ and $f = 0$ on $X \sim U$. Then $f|_{X \sim \bar{U}} \in A|_{X \sim \bar{U}}$ and $f|_{W \in A|_{W}}$. As $X = W \cup X \sim \bar{U}$, by the lemma $f \in A$. Thus $E_X \subset X \sim U$ by the defining property of the essential set.

Using the characterization obtained for E_M , Mullins goes on to show that A cannot restrict to all continuous functions on each of a countable number of closed sets whose union is $M(A)$ (metrizable). We use (3.2) to show some slightly stronger results.

DEFINITION. Let $p \in M(A)$. A *Jensen measure* μ_p for p on X is a representing measure for p on X satisfying $\log |f(p)| \leq \log \int |f| d\mu_p$ for all $f \in A$. Bishop has shown that any point $p \in M(A)$ always has a Jensen measure on X .

It follows that if F is a closed set in X and $A|_F$ is closed and if μ_p is a Jensen measure with $\mu_p(F) > 0$ then $p \in M(A|_F)$. In particular a Jensen measure always has either perfect closed support or is a point-mass.

COROLLARY 3.3. *If A is a function algebra on $X = \bigcup_{i=1}^{\infty} F_i$ where F_i are closed and $A|_{F_i} = C(F_i)$ for each i , then $A = C(X)$.*

PROOF. First A is approximately normal on X . For if $p \in M(A)$ and μ_p is a Jensen measure for p then $\mu_p(F_i) > 0$ for some i and thus μ_p is the point-mass at p . Thus for any closed set F in X , $M(A|_F) = F$. It follows that A is approximately normal.

To show $A = C(X)$ we show the essential set E of A on X is empty. But $E = \bigcup_{i=1}^{\infty} E \cap F_i$ and by the Baire category theorem one set $E \cap F_i$ has nonempty interior with respect to E . But this means that for some nonempty closed neighborhood \bar{V} in E , $A|_{\bar{V}} = C(\bar{V})$ and this contradicts the characterization of E given by (3.2).

We point out that the last part of the proof is exactly that used by Mullins in his original result. (3.3) has also been obtained by Mullins and by Gamlin and Wilken in a different fashion. As another application of (3.2) we obtain a slightly stronger result.

COROLLARY 3.4. *Let F be a subset of X which is closed and has no perfect subsets. If $X \sim F = \bigcup_{i=1}^{\infty} F_i$ where F_i are closed and $A|F_i = C(F_i)$ for all i , then $A = C(X)$.*

PROOF. If μ_p is any Jensen measure on X then either μ_p is supported in F or $\mu_p(F_i) > 0$ for some i . Since F contains no perfect subsets and since $A|F_i = C(F_i)$, in either case, μ_p must be the point-mass at p . It follows that $M(A) = X$ and that A is approximately normal on X .

By (3.3) if \bar{V} is a closed neighborhood in $X \sim F$ then $A|\bar{V} = C(\bar{V})$. Thus E , the essential set of A on X , is in F . Since $X = M(A)$, E is perfect. Thus E is empty and $A = C(X)$.

The last proof used some techniques from an elegant proof by T. Read of the result of Rudin (when each $F_i = \emptyset$).

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