AN EXTREMAL HARMONIC FUNCTION[1]

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Let Ω be a regular subregion of a locally Euclidean n-space V with Ω having a finite number of border components partitioned into two disjoint sets α and β (e.g. a spherical shell). Let h be the harmonic function which vanishes on α and has the constant value k on β such that $\int_\alpha (\partial h/\partial n)dS = 1$. The class of regular harmonic functions on Ω which are continuous on $\partial\Omega$, which vanish on α and which have been normalized so that $\int_\alpha |\partial u/\partial n| dS = 1$ will be denoted by $H_0(\Omega)$. For $u \in H_0(\Omega)$, the harmonic mean $m(u; k)$ is defined by $m^2(u; k) = \int_\beta u^2(\partial h/\partial n)dS$. We shall use $m(u; \lambda)$ for the mean over the level surface $\beta_\lambda = h^{-1}(\lambda)$ and $D(u; \lambda)$ for the Dirichlet integral over the region Ω_λ bounded by α and β_λ. The main result of this paper is the inequality: $\max h | \Omega_\lambda = m(h; \lambda) = D(h; \lambda) < m(u; \lambda) < D(u; \lambda)$, for $u \in H_0(\Omega)$, $u \neq \pm h$. Thus h minimizes the mean and the Dirichlet integral in the class $H_0(\Omega)$. This inequality can be used in the classification of locally Euclidean spaces. In particular, we shall show that $O_G = O_{H_0M} = O_{H_0D} = O_{H_0B}$. Here O_G is the class of spaces V possessing no Green's function, and O_{H_0K}, $K = M, D, B$ is the class of V on which there exist no nonconstant H_0K functions, that is, harmonic functions which vanish on the border of a boundary neighborhood of V and which are of finite mean, of finite Dirichlet integral, or bounded, respectively.

The methods employed here appear to be valid for arbitrary Riemannian n-spaces [1]. However, we shall assume that V is a locally Euclidean (flat) n-space [2].

Nevanlinna [3] proved the above inequality in two dimensions by the use of complex variables. It was later used by Sario [4] in classifying Riemann surfaces.

The proof of the inequality will be carried out by means of two lemmas and their corollaries.

LEMMA 1. *Let f be continuous on $\bar{\Omega}_\lambda$, the region bounded by α and β_λ. Then*

$$(1) \qquad \frac{d}{d\lambda}\int_{\Omega_\lambda} f dV = \int_{\beta_\lambda} \frac{f}{|\nabla h|} dS.$$

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PROOF. We express the volume integral as an iterated integral $\int f dV = \int\int f dS dt$, where t is measured normal to the level surfaces of h. To express this last integral in terms of h, we use $(dh/dt) = |\nabla h|$, or $dt = (dh/|\nabla h|)$, so that

$$\int_{\Omega_\lambda} f dV = \int_0^\lambda \int_{\beta_\lambda} \frac{f}{|\nabla h|} dS dh.$$

This iterated integral is now differentiated with respect to λ. The zeros of $|\nabla h|$ do not affect the integral on the right because they form a set of zero capacity (cf. Kellogg [5, p. 273]).

COROLLARY 1. *Let u be a function in $H_0(\Omega)$. Then*

$$\frac{1}{2}\frac{d}{d\lambda}\left[m^2(u;\lambda)\right] = \int_{\beta_\lambda} u \frac{\partial u}{\partial n} dS = D(u;\lambda). \tag{2}$$

PROOF. We first transform the integral for the mean to a volume integral, $m^2(u;\lambda) = \int u^2(\partial h/\partial n)dS = \int u^2 \nabla h \cdot n dS = \int 2u\nabla u \cdot \nabla h dV$. Then by Lemma 1,

$$\frac{d}{d\lambda}\left[m^2(u;\lambda)\right] = 2\int_{\beta_\lambda} u\nabla u \cdot \frac{\nabla h}{|\nabla h|} dS = 2\int_{\beta_\lambda} u \frac{\partial u}{\partial n} dS.$$

COROLLARY 2. *If $u \in H_0(\Omega)$, but $u \neq \pm h$, then*

$$D'(u;\lambda) > \left(\int_{\beta_\lambda} \left|\frac{\partial u}{\partial n}\right| dS\right)^2. \tag{3}$$

PROOF. We apply formula (1) to the Dirichlet integral:

$$D'(u;\lambda) = \frac{d}{d\lambda}\int_{\Omega_\lambda} |\nabla u|^2 dV = \int_{\beta_\lambda} \frac{|\nabla u|^2}{|\nabla h|} dS$$
$$= \int_{\beta_\lambda} \left(\frac{|\nabla u|}{|\nabla h|}\right)^2 |\nabla h| dS.$$

We then use the Schwarz inequality: $(\int fg|\nabla h|dS)^2 \leqq \int f^2|\nabla h|dS \cdot \int g^2|\nabla h|dS$, and obtain

$$D'(u;\lambda) \leqq \left(\int_{\beta_\lambda} \frac{|\nabla u|}{|\nabla h|} |\nabla h| dS\right)^2 \leqq \left(\int_{\beta_\lambda} \left|\frac{\partial u}{\partial n}\right| dS\right)^2. \tag{4}$$

Equality holds if and only if $|\nabla u|/|\nabla h| = t(\lambda)$ and $|\nabla u| = |\partial u/\partial n|$, which imply that $u = \pm h$.

COROLLARY 3. *If $u \in H_0(\Omega)$ and $u \neq \pm h$, then*

$$m'(u;0) > 1. \tag{5}$$

PROOF. On differentiating (2), we obtain $D'(u;\lambda)=[m'(u;\lambda)]^2+m(u;\lambda)m''(u;\lambda)$. For $\lambda=0$, this reduces to $D'(u;0)=[m'(u;0)]^2$. Equation (3) with $\lambda=0$ implies that $[m'(u;0)]^2=D'(u;0)>1$. Taking the square root of this yields (5) since $m(u;0)=0$ and $m(u;\lambda)\geqq 0$ so that $m'(u;0)\geqq 0$.

LEMMA 2. *If $u\neq \pm h$ is a function in $H_0(\Omega)$, then*

$$m''(u;\lambda) > 0. \tag{6}$$

PROOF. We apply the above Schwarz inequality with $f=u$, $g=(1/|\nabla h|)(\partial u/\partial n)$, to (2) and obtain

$$\begin{aligned} m^2(u;\lambda)m'^2(u;\lambda) &= \left(\int_{\beta_\lambda} u\frac{1}{|\nabla h|}\frac{\partial u}{\partial n}|\nabla h|\,dS\right)^2 \\ &\leqq \int_{\beta_\lambda} u^2|\nabla h|\,dS\int_{\beta_\lambda}\frac{1}{|\nabla h|^2}\left(\frac{\partial u}{\partial n}\right)^2|\nabla h|\,dS, \\ &\leqq m^2(u;\lambda)\int_{\beta_\lambda}\frac{|\nabla u|^2}{|\nabla h|}\,dS \leqq m^2(u;\lambda)D'(u;\lambda). \end{aligned}$$

However, from (2), $D'(u;\lambda)=m(u;\lambda)m''(u;\lambda)+m'^2(u;\lambda)$. Substituting this in the above inequality and simplifying, we get $0\leqq m^3(u;\lambda)m''(u;\lambda)$. As before equality holds if and only if $u=\pm h$.

We can now prove the main inequality. From (5) and (6), we see that

$$m'(u;\lambda) = \int_0^\lambda m''(u;\lambda)d\lambda + m'(u;0) > m'(u;0) > 1.$$

This together with equation (2) yields $D(u;\lambda)>m(u;\lambda)$. Since $\int(\partial h/\partial n)dS=1$ on β_λ, $D(h;\lambda)=m(h;\lambda)=\lambda$. Thus we have the desired result:

$$\begin{aligned} D(u;\lambda) > m(u;\lambda) &= \int_0^\lambda m'(u;\lambda)d\lambda \\ &> \int_0^\lambda d\lambda = \lambda = D(h;\lambda) = m(h;\lambda). \end{aligned} \tag{7}$$

We shall now use this inequality to prove that the classes of Euclidean spaces defined in the introduction to this paper coincide.

THEOREM. *For locally Euclidean n-spaces,*

$$O_G = O_{H_0K}, \qquad K = B, M, \text{ or } D.$$

PROOF. Let W be a regular subregion of V with border α. We consider an arbitrary regular region W', with border β, which contains W. We set $\Omega = W' - W$ and use the notation and terminology previously introduced. If $V \notin O_{H_0B}$, then there is a W and u such that $|u| \leqq K$ on $V-W$. We then have

$$m^2(u; k) = \int_\beta u^2 \frac{\partial h}{\partial n} dS \leqq K^2 \int_\beta \frac{\partial h}{\partial n} dS = K^2.$$

Thus $m(u; k)$ is uniformily bounded for every such W' and $V \notin O_{H_0M}$.

To prove the opposite inclusion, let $u \in H_0M(V-W)$. Then for $\Omega = W' - W$, $m(u; k) \leqq M$. The maximum principal together with the inequality (7) imply that $|h(z)| \leqq k \leqq M$. These harmonic functions are therefore uniformily bounded and there exists a sequence of them which converges to a harmonic function h_0 on $V-W$. Since each $|h(z)| \leqq M$, $|h_0(z)| \leqq M$ and $h_0 \in H_0B(V-W)$. Inequality (7) also shows that $h_0 \in H_0D(V-W)$.

That $H_0D(V-W) \subset H_0M(V-W)$ is an immediate consequence of (7) since the Dirichlet integral is a bound for the mean.

The harmonic measure ω_Ω of β with respect to Ω is the harmonic function with $\omega_\Omega | \alpha = 0$ and $\omega_\Omega | \beta = 1$. The harmonic measure of the ideal boundary of V is, by definition, $\omega = \lim_{W' \to V} \omega_\Omega$.

We shall finish the proof of the theorem by showing that $O_G = O_{H_0B}$. If $V \notin O_G$ then $\omega \in H_0B(V-W)$ [6] and $V \notin O_{H_0B}$. If $V \notin O_{H_0B}$, let $|u| \leqq K$ on $V-W$. For every $\Omega = W' - W$, $m(u; k) \leqq K$. The harmonic measure ω_Ω is related to h_Ω by $\omega_\Omega = h_\Omega / k_\Omega$. Since $k_\Omega \leqq K$, $\lim k_\Omega \leqq K$. It follows that $\omega = \lim(h_\Omega / k_\Omega) \neq 0$ and therefore $V \notin O_G$ [6].

REMARK. This theorem provides a partial solution to problem (1), page 109 of Sario [7].

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