A SHORT PROOF OF A THEOREM OF L. JANOS

MICHAEL EDELSTEIN[1]

THEOREM (JANOS). *Let X be a compact metrizable topological space and $f: X \to X$ a continuous one-to-one mapping with $\bigcap_1^\infty f^n[X]$ a singleton. Given λ, $0<\lambda<1$, there exists a metric ρ on X such that the metric topology of (X, ρ) is identical with the original one and $\rho(f(x), f(y)) = \lambda\rho(x, y)$ for all $x, y \in X$.*

It is the purpose of this note to provide a proof of a somewhat stronger result, which seems to be shorter and simpler than any of those outlined by Janos in [1], [2] and [3].

THEOREM. *Let X be a compact metrizable topological space and $f: X \to X$ a continuous one-to-one mapping with $\bigcap_1^\infty f^n[X] = \{x_0\}$, where $x_0 \in X$. Given λ, $0<\lambda<1$, a homeomorphism h of X into l_2 exists such that*

$$\|h(f(x')) - h(f(x''))\| = \lambda\|h(x') - h(x'')\| \qquad \text{for all } x', x'' \in X.$$

PROOF. We may assume that $X \sim \{x_0\} \neq \emptyset$. Let $\mathcal{B}$ be a countable base for (the open nonempty set) $X \sim f[X]$. To each pair $U, V \in \mathcal{B}$, such that $\overline{U} \subset V$, we make correspond a continuous $\phi: X \to [0, 1]$ such that $\phi[U] = 1$ and $\phi[X \sim V] = 0$. Using the odd positive integers as an index set we obtain the family $\{\phi_{2n-1}: n = 1, 2, \cdots\}$ of mappings. Since $f[X]$ is closed and $\phi_{2n-1}f^{-1}: f[X] \to [0, 1]$ is continuous for each $n = 1, 2, \cdots$ there exists, by the Tietze extension theorem, a continuous $\phi_{2(2n-1)}: X \to [0, 1]$ which coincides with $\phi_{2n-1}f^{-1}$ on $f[X]$. Thus $\phi_{2(2n-1)}(f(x)) = \phi_{2n-1}(x)$ ($n = 1, 2, \cdots; x \in X$). Assuming $\phi_{2^{m-1}(2n-1)}: X \to [0, 1]$ defined, continuous and satisfying

$$\phi_{2^{m-1}(2n-1)}(f(x)) = \phi_{2^{m-2}(2n-1)}(x) \qquad (n = 1, 2, \cdots; x \in X),$$

we define $\phi_{2^m(2n-1)}: X \to [0, 1]$ by choosing a continuous extension of $\phi_{2^{m-1}(2n-1)}f^{-1}: f[X] \to [0, 1]$ to the whole of X, thereby obtaining continuous mappings $\phi_{2^m(2n-1)}$ of X into $[0, 1]$ for all $m = 0, 1, \cdots$, $n = 1, 2, \cdots$ and satisfying

$$\phi_{2^m(2n-1)}(f(x)) = \phi_{2^{m-1}(2n-1)}(x)$$
$$(m = 1, 2, \cdots; n = 1, 2, \cdots, x \in X).$$

Also, clearly,

Received by the editors February 12, 1968.

[1] Research supported by N.R.C. Grant A-3999.

$$\phi_{(2n-1)}(f(x)) = 0 \qquad (n = 1, 2, \cdots ; x \in X).$$

We now define h as follows. If $k = 2^m(2n-1)$ set

$$y_k = \lambda^{m+n}\phi_{2^m(2n-1)}(x) \quad \text{and} \quad h(x) = (y_1, y_2, \cdots, y_k, \cdots).$$

Obviously $h(x) \in l_2$. It is a straightforward matter to verify that h is one-to-one and continuous; hence, by compactness of X, a homeomorphism onto $h[X]$. Finally,

$$\begin{aligned} &\|h(f(x')) - h(f(x''))\|^2 \\ &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \lambda^{2(m+n)}[\phi_{2^m(2n-1)}(f(x')) - \phi_{2^m(2n-1)}(f(x''))]^2 \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \lambda^{2(m+n)}[\phi_{2^{m-1}(2n-1)}(x') - \phi_{2^{m-1}(2n-1)}(x'')]^2 \\ &= \lambda^2 \sum_{m=0}^{\infty} \sum_{n=1} \lambda^{2(m+n)}[\phi_{2^m(2n-1)}(x') - \phi_{2^m(2n-1)}(x'')]^2 \\ &= \lambda^2\|h(x') - h(x''))\|^2 \end{aligned}$$

and the theorem follows.

References

1. L. Janos, *A converse of the Banach theorem in the case of one to one contracting mappings*, Notices Amer. Math. Soc. **11** (1964), 686.

2. ———, *Homothetic property of contractive one-to-one mappings*, Notices Amer. Math. Soc. **13** (1966), 818.

3. ———, *One-to-one contractive mappings on compact spaces*, Notices Amer. Math. Soc. **14** (1967), 133.

Dalhousie University