SOME WHITEHEAD PRODUCTS ON ODD SPHERES

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1. **Introduction.** We consider the Whitehead product $[\iota_n, \iota_n] \in \Pi_{2n-1}(S^n)$ where ι_n is a generator of $\Pi_n(S^n)$. For n even, it is well known that $[\iota_n, \iota_n]$ generates an infinite cyclic subgroup of $\Pi_{2n-1}(S^n)$. Furthermore, this subgroup is actually a direct summand unless $n=2$, 4 or 8. For n odd, $[\iota_n, \iota_n]$ generates a subgroup of order two. In [3], G. V. Krishnarao recently showed that, for $S>0$ and $n=4S+1$, the subgroup actually splits off as a direct summand. In this note we show this to be true for the class of integers n such that $n+1\neq 2^r$. This result is also known to James [2].

In what follows (Π, n) will denote an Eilenberg-MacLane space such that $\Pi_i((\Pi, n))=0$ for $i\neq n$ and $\Pi_n((\Pi, n))=\Pi$. Π will always be a cyclic group. α_n will denote a generator of $\Pi_n((\Pi, n))=\Pi_n(\Pi, n)$ and X will be the space $S^n\cup_g e^{2n}$ where $g=[\iota_n, \iota_n]$.

It is a classically known fact that the kernel of Σ: $\Pi_{2n-1}(S^n)\to\Pi_{2n}(S^{n+1})$ is the subgroup generated by $[\iota_n, \iota_n]$. We thus have the following short exact sequence for odd n,

$$0 \to Z_2 \to \Pi_{2n-1}(S^n) \to \Pi_{2n}(S^{n+1}) \to 0.$$

In order to split this sequence, all we need to do is find a homomorphism f: $\Pi_{2n-1}(S^n)\to Z_2$ which is nonzero on $[\iota_n, \iota_n]$. Once the sequence is split we have $\Pi_{2n-1}(S^n)=Z_2\oplus\Pi^S_{n-1}$ where Π^S_{n-1} is the $(n-1)$st stable stem of homotopy groups of spheres.

2. **The result.** For $n+1\neq 2^r$, there is an Adem relation of the form $\mathrm{Sq}^{n+1}=\sum_{i=1}^k a_i b_i$ where a_i and b_i are elements in the (mod 2) Steenrod Algebra [4, p. 2]. We need the following

LEMMA. *For n odd, $n+1\neq 2^r$, $\mathrm{Sq}^{n+1}=\sum_{i=1}^k a_i b_i$ as above, and no $a_i=\mathrm{Sq}^1$.*

PROOF. We only have to rule out the term $\mathrm{Sq}^1\mathrm{Sq}^n$ since any term such as $\mathrm{Sq}^1(\mathrm{Sq}^p\mathrm{Sq}^r)$ can be regrouped as $(\mathrm{Sq}^{p+1})\mathrm{Sq}^r$ to serve our ends. For this purpose, notice [4, p. 11] that $\{\mathrm{Sq}^{2^i}\}$ generates the (mod 2) Steenrod Algebra, as an algebra. Thus Sq^n, n odd, can be broken up as

$$\sum_{a_i+b_i=n} \mathrm{Sq}^{a_i}\mathrm{Sq}^{b_i}, \quad \text{where } n>\deg b_i>0.$$

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So

$$\mathrm{Sq}^1\mathrm{Sq}^r = \sum_{a_i+b_i=n} \mathrm{Sq}^{a_i+1}\mathrm{Sq}^{b_i}$$

and the lemma is proved.

The above Adem relation may thus be regrouped in such a way that no $a_i = \mathrm{Sq}^1$. We now use this Adem relation to construct a Postnikov tower which is a universal example for the secondary cohomology operation associated to our particular grouping in the Adem relation. The operation is a modulo 2 operation defined on integral cohomology classes. In the diagram below, deg b_i refers to the ordinary degree of b_i as an element in the Steenrod Algebra.

$$\begin{array}{ccccc}
 & (Z_2, 2n-1) \xrightarrow{i_2} & E_2 & & \\
 & & \downarrow P_2 & & \\
\prod_{i=1}^{k} (Z_2, n+\deg b_i - 1) & \xrightarrow{i_1} & E_1 & \xrightarrow[\Phi]{} & (Z_2, 2n) \\
 & & \downarrow P_1 & & \\
 & & (Z, n) & \xrightarrow[\times_{i=1}^{k} b_i]{} & \prod_{i=1}^{k} (Z_2, n+\deg b_i)
\end{array}$$

Consider the homotopy groups of the total space E_2. $\Pi_n(E_2) \supset Z$ on a generator η such that $p_{1\#}(p_{2\#}(\eta)) = \alpha_n \in \Pi_n(Z, n)$ and $\Pi_{2n-1}(E_2) = Z_2$ by some diagram-chasing together with a check on the integers deg b_i using Lemma 1.

The secondary operations, Φ, defined above are particular examples of a class of operations considered in [1] by Brown and Peterson, where the following is proved:

THEOREM. *If* $u \in H^n(X; Z)$ *is a generator and* $\Phi(u) \neq 0$, *then* $[\eta, \eta] \neq 0$.

Brown and Peterson then show that, for the sort of Φ constructed above, $\Phi(u) \neq 0$.

Now let $S^n \xrightarrow{f'} (Z, n)$ represent α_n. Trivially this map lifts to a map $S^n \xrightarrow{f} E_2$ which represents η. But the naturality of Whitehead products gives us $f_\#([\iota_n, \iota_n]) = [f_\#(\iota_n), f_\#(\iota_n)] = [\eta, \eta] \neq 0$, so that we have

THEOREM 1. *There is a space* E_2 *and a map* $f: S^n \rightarrow E_2$ *so that the induced map on homotopy groups takes* $[\iota_n, \iota_n] \in \Pi_{2n-1}(S^n)$ *to a nonzero element in* $\Pi_n(E_2) = Z_2$, *when* $n+1 \neq 2^r$ *and* n *is odd.*

This splits the sequence in the introduction and establishes the claim made there.

References

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4. N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*, Princeton Univ. Press, Princeton, N. J., 1962.

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