A THEOREM ON A MAPPING FROM A SPHERE TO THE CIRCLE AND THE SIMULTANEOUS DIAGONALIZATION OF TWO HERMITIAN MATRICES

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1. Introduction and statement of the theorems. We denote by F the field R of real numbers, the field C of complex numbers, or the field H of real quaternions, and by F^n an n-dimensional left vector space over F. If A is a matrix with elements in F, we denote by A^* its conjugate transpose. In all three cases of F, an $n\times n$ matrix A is said to be *hermitian* if $A=A^*$ and *unitary* if $AA^*=I$, where I is the $n\times n$ identity matrix. An $n\times n$ hermitian matrix A is said to be *positive definite* if $uAu^*>0$ for all u ($\neq 0$) in F^n. Here and in what follows we regard u as a $1\times n$ matrix and identify a 1×1 matrix with its single element.

The purpose of this note is to prove Theorem 1 on a mapping from a sphere to the circle, and use it to prove Theorem 2 on the simultaneous diagonalization of two hermitian matrices.

THEOREM 1. *Let A and B be two $n\times n$ hermitian matrices with elements in F such that $(uAu^*)^2+(uBu^*)^2>0$ for all u ($\neq 0$) in F^n, and $S(F^n)$ the unit sphere in F^n (i.e. $S(F^n)=\{u\in F^n: uu^*=1\}$). If $F=R$ and $n\geqq 3$ or $F=C$ or H and $n\geqq 2$, then the image of the mapping $f: S(F^n)\to S(R^2)$ defined by*

$$f(u)=\left(\frac{uAu^*}{((uAu^*)^2+(uBu^*)^2)^{1/2}},\ \frac{uBu^*}{((uAu^*)^2+(uBu^*)^2)^{1/2}}\right)$$

is a closed circular arc of length $<\pi$.

THEOREM 2. *Let A and B be two $n\times n$ hermitian matrices with elements in F such that $(uAu^*)^2+(uBu^*)^2>0$ for all u ($\neq 0$) in F^n. If $F=R$ and $n\geqq 3$ or $F=C$ or H and $n\geqq 2$, then A and B can be diagonalized simultaneously (i.e. there exists a nonsingular $n\times n$ matrix U with elements in F such that UAU^* and UBU^* are diagonal matrices).*

For the case $F=R$ and $n\geqq 3$, Theorem 2 has been proved by Greub [1, pp. 231–237], and Calabi [2][2] by different methods.

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2. A mapping from a sphere to the unit circle. We first prove a lemma.

LEMMA 1. *For any real numbers* $a_0, \cdots, a_4, b_0, \cdots, b_4$ *the following system of equations*

$$(1)\qquad \begin{aligned} a_0(x^2 - y^2) + a_1xy + a_2xz + a_3yz + a_4z^2 &= 0, \\ b_0(x^2 - y^2) + b_1xy + b_2xz + b_3yz + b_4z^2 &= 0, \end{aligned}$$

has nontrivial real solutions.

PROOF. There are two cases.

Case 1. $a_0b_1 - a_1b_0 = 0$. In this case we put $z=0$ in (1), and (1) becomes

$$(2)\qquad a_0(x^2 - y^2) + a_1xy = 0, \qquad b_0(x^2 - y^2) + b_1xy = 0.$$

Since $a_0b_1 - a_1b_0 = 0$, (2) has nontrivial real solutions. Hence (1) has nontrivial real solutions.

Case 2. $a_0b_1 - a_1b_0 \neq 0$. In this case we put $z=1$ in (1), and (1) becomes

$$(3)\qquad \begin{aligned} a_0(x^2 - y^2) + a_1xy + a_2x + a_3y + a_4 &= 0, \\ b_0(x^2 - y^2) + b_1xy + b_2x + b_3y + b_4 &= 0. \end{aligned}$$

Since $a_0b_1 - a_1b_0 \neq 0$, (3) is equivalent to

$$(4)\quad x^2 - y^2 + a_2'x + a_3'y + a_4' = 0, \qquad xy + b_2'x + b_3'y + b_4' = 0,$$

where a_2', a_3', a_4', b_2', b_3' and b_4' are some real numbers. But (4) obviously has real solutions. Hence (1) has nontrivial real solutions.

We now prove Theorem 1. Since f is continuous and $S(F^n)$ is connected and compact, $f(S(F^n))$ is a closed circular arc. Therefore, Theorem 1 will be proved if we can show that, for any (a, b) in $S(R^2)$, (a, b) and $(-a, -b)$ cannot both belong to $f(S(F^n))$.

Assume that there exists (a_0, b_0) in $S(R^2)$ such that (a_0, b_0) and $(-a_0, -b_0)$ both belong to $f(S(F^n))$. Then by the definition of f there exist u_1 and u_2 in $F^n \setminus \{0\}$ such that $(a_0, b_0) = (u_1Au_1^*, u_1Bu_1^*)$ and $(-a_0, -b_0) = (u_2Au_2^*, u_2Bu_2^*)$. Obviously, u_1 and u_2 are linearly independent over F. Since $F=R$ and $n \geqq 3$ or $F=C$ or H and $n \geqq 2$, there exists u_3 in F^n such that u_1, u_2 and u_3 are linearly independent over R. Now, for any (x, y, z) in R^3 we have

$$\begin{aligned} &(xu_1 + yu_2 + zu_3)A(xu_1 + yu_2 + zu_3)^* \\ &\qquad = a_0(x^2 - y^2) + a_1xy + a_2xz + a_3yz + a_4z^2, \end{aligned}$$

and

$$(xu_1 + yu_2 + zu_3)B(xu_1 + yu_2 + zu_3)^* = b_0(x^2 - y^2) + b_1xy + b_2xz + b_3yz + b_4z^2,$$

where the a's and b's are all real numbers; for example, $a_1 = u_1Au_2^* + u_2Au_1^*$, $b_1 = u_1Bu_2^* + u_2Bu_1^*$. Therefore, by Lemma 1, there exists $(x_0, y_0, z_0) \neq (0, 0, 0)$ in R^3 such that

$$(x_0u_1 + y_0u_2 + z_0u_3)A(x_0u_1 + y_0u_2 + z_0u_3)^* = 0,$$

and

$$(x_0u_1 + y_0u_2 + z_0u_3)B(x_0u_1 + y_0u_2 + z_0u_3)^* = 0.$$

Since u_1, u_2 and u_3 are linearly independent over R, we obtain a contradiction to the hypothesis that $(uAu^*)^2 + (uBu^*)^2 > 0$ for all $u(\neq 0)$ in F^n. Hence Theorem 1 is proved.

3. Simultaneous diagonalization of two hermitian matrices. Suppose that the conditions of Theorem 2 are satisfied. By Theorem 1, $f(S(F^n))$ is a closed circular arc of length $<\pi$. Let (a, b) be the midpoint of this circular arc. Then if u is any point in $S(F^n)$ and if θ $(<\pi/2)$ is the angle between the radii of $S(R^2)$ with end points (a, b) and $f(u)$, we have

$$a \frac{uAu^*}{((uAu^*)^2 + (uBu^*)^2)^{1/2}} + b \frac{uBu^*}{((uAu^*)^2 + (uBu^*)^2)^{1/2}} = \cos \theta > 0.$$

Therefore, $aA + bB$ is positive definite, and Theorem 2 is proved by the following lemma:

LEMMA 2. *If A and B are two $n \times n$ hermitian matrices with elements in F such that $aA + bB$ is positive definite for some (a, b) in R^2, then A and B can be diagonalized simultaneously.*

PROOF. Since $aA + bB$ is a positive definite hermitian matrix, one of the a and b, say a, is not zero and there exists a unitary matrix U_1 such that

$$U_1(aA + bB)U_1^* = \operatorname{diag}(a_1, \cdots, a_n),$$

where $a_1, \cdots, a_n$ are positive real numbers. (This is well known if $F = R$ or C; for example, see [**3**, pp. 12–13]. For $F = H$, it is proved in [**4**] and [**5**].) Let $U_2 = \operatorname{diag}(1/\sqrt{a_1}, \cdots, 1/\sqrt{a_n})$. Then

$$U_2U_1(aA + bB)U_1^*U_2^* = I, \tag{5}$$

where I is the $n \times n$ identity matrix. Since $U_2U_1BU_1^*U_2^*$ is a hermitian matrix, there exists a unitary matrix U_3 such that

(6) $UBU^* = \text{diagonal matrix}, \quad \text{where} \quad U = U_3U_2U_1.$

From (5) and (6) it follows that

$$UAU^* = 1/a(I - bUBU^*) = \text{diagonal matrix}.$$

Thus Lemma 2 is proved.

ADDED IN PROOF. The author has just learned that, for the case F = real closed field and $n \geqq 3$, Theorem 2 has been proved by Wonenburger [J. Math. Mech. **15** (1966), 617–622]; and for the case $F = R$ and $n \geqq 3$ or $F = C$ and $n \geqq 2$ by Kraljević [Glasnik Mat. Ser. III **1** (**21**) (1966), 57–63]. Their methods of proof are quite different from that of the author.

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