AN INEQUALITY FOR A STEKLOFF EIGENVALUE BY THE METHOD OF DEFECT[1]

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1. **Introduction.** In this paper we give a lower bound to the first nonzero eigenvalue p_2 of the Stekloff problem [8] for plane regions having two axes of symmetry. Such bounds lead to a priori inequalities which are useful in giving error bounds for approximate solutions to the Neumann problem for Poisson's equation (see [5, §2]).

In [9], Weinstock has given an isoperimetric upper bound for p_2, but good lower bounds, which are more useful, are more elusive (see [1], [4], [6]).

We first prove a nodal-line theorem of some interest in itself. This nodal-line theorem does not seem to appear anywhere in the literature, although its proof is a straightforward application of familiar methods (see [7]). The results of the nodal-line theorem then permit us to use the method of defect (see, e.g., [2]) to obtain our bound by integrating an easily obtained one-dimensional version of the desired inequality.

2. **Preliminaries.** Let B be a bounded, connected domain of the x_1, x_2-plane with piecewise smooth boundary ∂B. We consider the Stekloff eigenvalue problem

$$\Delta u = 0 \text{ in } B, \qquad \partial u/\partial n = pu \text{ on } \partial B, \tag{1}$$

where Δ is the Laplacian and n the unit outer normal on ∂B. The problem has a discrete spectrum of eigenvalues $0 = p_1 < p_2 \leqq p_3 \leqq \cdots$ with corresponding eigenfunctions $u_1 = \text{constant}$, u_2, u_3, $\cdots$. The eigenvalues can be characterized by

$$p_n = \min \frac{\int_B |\operatorname{grad} v|^2 dx}{\oint_{\partial B} v^2 ds}, \tag{2}$$

where the minimum is taken over all continuous and piecewise continuously differentiable functions v satisfying

$$\oint_{\partial B} v u_k ds = 0, \qquad k = 1, 2, \cdots, n-1. \tag{3}$$

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The minimum of (2) is attained when and only when v is an eigenfunction of (1) associated with p_n.

3. **The nodal-line theorem.** A curve in B along which an eigenfunction u_n vanishes is called a nodal line of u_n. What we prove is

THEOREM. *The nodal lines of u_n divide B into no more than n subdomains, and no nodal line is a closed curve.*

PROOF. We prove the second part of the theorem first. If a nodal line were closed, we would have $u_n \equiv 0$ in the interior of the curve since $\Delta u_n = 0$ in B. But then u_n would vanish identically in B by the Unique Continuation Theorem for harmonic functions (see, e.g., [**3**, Chapter X, §5]). Thus no nodal line can be a closed curve.

Now suppose that the nodal lines of u_n divide B into more than n subdomains. Let $D_1, D_2, \cdots, D_n$ be n of these subdomains. Note that $\partial B \cap \partial D_i$ is not empty. Define w_i to agree with u_n on D_i and vanish on $B - D_i$, $i = 1, 2, \cdots, n$. Notice that $w_i \not\equiv 0$ on $\partial B \cap \partial D_i$, otherwise $w_i \equiv 0$ in D_i (since $\Delta w_i = 0$ in D_i). Thus, we can find a linear combination, say $v = \sum_{i=1}^n a_i w_i$, such that

$$\oint_{\partial B} v^2 ds = 1$$

and v satisfies (3). Moreover, the w_i, hence v, are continuous and piecewise continuously differentiable (see [**3**, Chapter X, §9]). Since $\partial v / \partial n = p_n v$ on ∂B, it follows from Green's first identity that

$$\frac{\int_B |\operatorname{grad} v|^2 dx}{\oint_{\partial B} v^2 ds} = p_n.$$

Thus v minimizes (2) and is therefore an eigenfunction of (1). Hence v is harmonic and vanishes on the nonempty subdomain $B - D_1 \cup \cdots \cup D_n$. Using the Unique Continuation Theorem we arrive at a contradiction. This completes the theorem.

The application we wish to make of this theorem is the following. Since an eigenfunction u_2 associated with p_2 satisfies $\oint_{\partial B} u_2 \, ds = 0$, we see that u_2 must have a nodal line. By the theorem u_2 cannot have more than one, hence has exactly one.

4. **The method of defect.** Suppose now our region B has two distinct axes of symmetry. They may be assumed perpendicular, and, with no loss of generality, we take them to be the x_1 and x_2 axes. We will call a function defined on B even-even, odd-odd, even-odd, or odd-even as u is respectively even in both x_1 and x_2, odd in both x_1

and x_2, even in x_1 and odd in x_2, or odd in x_1 and even in x_2. Every eigenfunction of (1) can be assumed to belong to one of these *symmetry classes*. From the previous section, u_2 must be in the even-odd or odd-even symmetry classes. (Otherwise, u_2 has an even number of nodal lines.) Hence, the nodal line is an axis of symmetry.

The one-dimensional inequality

$$[u(0)]^2 + [u(l)]^2 \leqq \frac{l}{2}\int_0^l [u'(y)]^2 dy, \tag{4}$$

for continuous and piecewise continuously differentiable functions on the interval $(0, l)$ which satisfy $u(0) = -u(l)$, is easily shown by solving the Euler equation.

Let us first consider the case when the eigenfunction u_2 is odd across the x_1-axis. Suppose the boundary ∂B can be expressed by $x_2 = \pm f_1(x_1)$, $-a_1 \leqq x_1 \leqq a_1$. Then, employing (4),

$$\begin{aligned}
\oint_{\partial B} u_2^2 ds
&= 2\int_{-a_1}^{a_1} [u_2(x_1, f_1(x_1))]^2 (1 + [f_1'(x_1)]^2)^{1/2} dx_1 \\
&\leqq \int_{-a_1}^{a_1} (1 + [f_1'(x_1)]^2)^{1/2} \left[f_1(x_1) \int_{-f_1(x_1)}^{f_1(x_1)} \left[\frac{\partial u_2(x_1, x_2)}{\partial x_2}\right]^2 dx_2 \right] dx_1 \\
&\leqq \left[\max_{-a_1 \leqq x_1 \leqq a_1} f_1(x_1)(1 + [f_1'(x_1)]^2)^{1/2} \right] \int_{-a_1}^{a_1} \int_{-f_1(x_1)}^{f_1(x_1)} \left(\frac{\partial u_2}{\partial x_2}\right)^2 dx_2 dx_1 \\
&\leqq \left[\max_{-a_1 \leqq x_1 \leqq a_1} f_1(x_1)(1 + [f_1'(x_1)]^2)^{1/2} \right] \int_B |\operatorname{grad} u_2|^2 dx.
\end{aligned}$$

By similarly treating the case when u_2 is odd across the x_2-axis, we have our inequality:

$$p_2^{-1} \leqq \max_{i=1,2} \left[\max_{-a_i \leqq x_i \leqq a_i} f_i(x_i)(1 + [f_i'(x_i)]^2)^{1/2} \right], \tag{5}$$

where $x_1 = \pm f_2(x_2)$, $-a_2 \leqq x_2 \leqq a_2$, is another representation of ∂B.

5. **Some examples.** We give a few examples of the application of (5) to particular regions. For an upper bound, we use the isoperimetric inequality of Weinstock [9], which says

$$p_2 \leqq 2\pi/L, \tag{6}$$

where L is the length of ∂B.

First, consider the rhombus

$$|x_1/a_1| + |x_2/a_2| \leqq 1.$$

We have $L=4(a_1^2+a_2^2)^{1/2}$ and

$$f_i(x_i) = a_j(1 - |x_i/a_i|), \qquad j \neq i.$$

Thus, from (5) and (6), we have

$$\min(a_2/a_1,\, a_1/a_2) \leqq p_2(a_1^2 + a_2^2)^{1/2} \leqq \pi/2. \tag{7}$$

When $a_1=a_2=S/\sqrt{2}$, the rhombus is a square of side S, and (7) becomes $1\leqq p_2S\leqq\pi/2$, whereas the exact value to five places is $p_2S=1.3765\cdots$.

Next, consider the ellipse $(x_1/a_1)^2+(x_2/a_2)^2\leqq 1$ for which

$$f_i(x_i) = a_j(1 - (x_i/a_i)^2)^{1/2}, \qquad j \neq i.$$

For simplicity, we worsen (6) by combining it with the classical isoperimetric inequality

$$L^2 \geqq 4\pi A, \tag{8}$$

where A, the area of the ellipse, is πa_1a_2. Thus, we have

$$[\max(a_1, a_2)]^{-1} \leqq p_2 \leqq (a_1a_2)^{-1/2}. \tag{9}$$

For the special case of a disc when $a_1=a_2=R$, we attain equality on both sides and $p_2R=1$.

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