1. Introduction. In this paper we give a lower bound to the first nonzero eigenvalue $p_2$ of the Stekloff problem [8] for plane regions having two axes of symmetry. Such bounds lead to a priori inequalities which are useful in giving error bounds for approximate solutions to the Neumann problem for Poisson's equation (see [5, §2]).

In [9], Weinstock has given an isoperimetric upper bound for $p_2$, but good lower bounds, which are more useful, are more elusive (see [1], [4], [6]).

We first prove a nodal-line theorem of some interest in itself. This nodal-line theorem does not seem to appear anywhere in the literature, although its proof is a straightforward application of familiar methods (see [7]). The results of the nodal-line theorem then permit us to use the method of defect (see, e.g., [2]) to obtain our bound by integrating an easily obtained one-dimensional version of the desired inequality.

2. Preliminaries. Let $B$ be a bounded, connected domain of the $x_1, x_2$-plane with piecewise smooth boundary $\partial B$. We consider the Stekloff eigenvalue problem

\[ (1) \quad \Delta u = 0 \text{ in } B, \quad \frac{\partial u}{\partial n} = pu \text{ on } \partial B, \]

where $\Delta$ is the Laplacian and $n$ the unit outer normal on $\partial B$. The problem has a discrete spectrum of eigenvalues $0 = p_1 < p_2 \leq p_3 \leq \cdots$ with corresponding eigenfunctions $u_1 = \text{constant}, \ u_2, \ u_3, \ \cdots$. The eigenvalues can be characterized by

\[ (2) \quad p_n = \min \frac{\int_B |\nabla v|^2 dx}{\int_{\partial B} v^2 ds}, \]

where the minimum is taken over all continuous and piecewise continuously differentiable functions $v$ satisfying

\[ (3) \quad \int_{\partial B} vu_k ds = 0, \quad k = 1, 2, \cdots, n - 1. \]
The minimum of (2) is attained when and only when \( v \) is an eigenfunction of (1) associated with \( p_n \).

3. The nodal-line theorem. A curve in \( B \) along which an eigenfunction \( u_n \) vanishes is called a nodal line of \( u_n \). What we prove is

**Theorem.** The nodal lines of \( u_n \) divide \( B \) into no more than \( n \) subdomains, and no nodal line is a closed curve.

**Proof.** We prove the second part of the theorem first. If a nodal line were closed, we would have \( u_n \equiv 0 \) in the interior of the curve since \( \Delta u_n = 0 \) in \( B \). But then \( u_n \) would vanish identically in \( B \) by the Unique Continuation Theorem for harmonic functions (see, e.g., [3, Chapter X, §5]). Thus no nodal line can be a closed curve.

Now suppose that the nodal lines of \( u_n \) divide \( B \) into more than \( n \) subdomains. Let \( D_1, D_2, \ldots, D_n \) be \( n \) of these subdomains. Note that \( \partial B \cap \partial D_i \) is not empty. Define \( w_i \) to agree with \( u_n \) on \( D_i \) and vanish on \( B - D_i \), \( i = 1, 2, \ldots, n \). Notice that \( w_i \equiv 0 \) on \( \partial B \cap \partial D_i \), otherwise \( w_i \equiv 0 \) in \( D_i \) (since \( \Delta w_i = 0 \) in \( D_i \)). Thus, we can find a linear combination, say \( v = \sum_{i=1}^{n} a_i w_i \), such that

\[
\oint_{\partial B} v^2 ds = 1
\]

and \( v \) satisfies (3). Moreover, the \( w_i \), hence \( v \), are continuous and piecewise continuously differentiable (see [3, Chapter X, §9]). Since \( \partial v / \partial n = p_n v \) on \( \partial B \), it follows from Green's first identity that

\[
\frac{\int_{\partial B} \text{grad } v \cdot 2dx}{\oint_{\partial B} v^2 ds} = p_n.
\]

Thus \( v \) minimizes (2) and is therefore an eigenfunction of (1). Hence \( v \) is harmonic and vanishes on the nonempty subdomain \( B - D_1 \cup \cdots \cup D_n \). Using the Unique Continuation Theorem we arrive at a contradiction. This completes the theorem.

The application we wish to make of this theorem is the following. Since an eigenfunction \( u_2 \) associated with \( p_2 \) satisfies \( \int_{\partial B} u_2 ds = 0 \), we see that \( u_2 \) must have a nodal line. By the theorem \( u_2 \) cannot have more than one, hence has exactly one.

4. The method of defect. Suppose now our region \( B \) has two distinct axes of symmetry. They may be assumed perpendicular, and, with no loss of generality, we take them to be the \( x_1 \) and \( x_2 \) axes. We will call a function defined on \( B \) even-even, odd-odd, even-odd, or odd-even as \( u \) is respectively even in both \( x_1 \) and \( x_2 \), odd in both \( x_1 \)
and \( x_2 \), even in \( x_1 \) and odd in \( x_2 \), or odd in \( x_1 \) and even in \( x_2 \). Every eigenfunction of (1) can be assumed to belong to one of these symmetry classes. From the previous section, \( u_2 \) must be in the even-odd or odd-even symmetry classes. (Otherwise, \( u_2 \) has an even number of nodal lines.) Hence, the nodal line is an axis of symmetry.

The one-dimensional inequality

\[
[u(0)]^2 + [u(l)]^2 \leq \frac{l}{2} \int_0^l [u'(y)]^2 dy,
\]

for continuous and piecewise continuously differentiable functions on the interval \((0, l)\) which satisfy \( u(0) = -u(l) \), is easily shown by solving the Euler equation.

Let us first consider the case when the eigenfunction \( u_2 \) is odd across the \( x_1 \)-axis. Suppose the boundary \( \partial B \) can be expressed by \( x_2 = f_1(x_1), \quad -a_1 \leq x_1 \leq a_1 \). Then, employing (4),

\[
\oint_{\partial B} u_2^2 \, ds = 2 \int_{-a_1}^{a_1} [u_2(x_1, f_1(x_1))]^2 (1 + [f'_1(x_1)]^2)^{1/2} \, dx_1
\]

\[
\leq \int_{-a_1}^{a_1} (1 + [f'_1(x_1)]^2)^{1/2} \left[ f_1(x_1) \int_{-f_1(x_1)}^{f_1(x_1)} \frac{\partial u_2(x_1, x_2)}{\partial x_2} \right]^2 \, dx_1
\]

\[
\leq \left[ \max_{-a_1 \leq x_1 \leq a_1} f_1(x_1) (1 + [f'_1(x_1)]^2)^{1/2} \right] \left[ \int_{-a_1}^{a_1} \int_{-f_1(x_1)}^{f_1(x_1)} \left( \frac{\partial u_2}{\partial x_2} \right)^2 \, dx_2 \, dx_1 \right]
\]

\[
\leq \left[ \max_{-a_1 \leq x_1 \leq a_1} f_1(x_1)(1 + [f'_1(x_1)]^2)^{1/2} \right] \int_B |\text{grad } u_2|^2 \, dx.
\]

By similarly treating the case when \( u_2 \) is odd across the \( x_2 \)-axis, we have our inequality:

\[
(p_2^{-1}) \leq \max_{i=1, 2} \left[ \max_{-a_i \leq x_i \leq a_i} f_i(x_i)(1 + [f'_i(x_i)]^2)^{1/2} \right],
\]

where \( x_i = \pm f_i(x_2) \), \( -a_2 \leq x_2 \leq a_2 \), is another representation of \( \partial B \).

5. Some examples. We give a few examples of the application of (5) to particular regions. For an upper bound, we use the isoperimetric inequality of Weinstock [9], which says

\[
p_2 \leq 2\pi/L,
\]

where \( L \) is the length of \( \partial B \).
First, consider the rhombus
\[ |x_i/a_i| + |x_j/a_j| \leq 1. \]
We have \( L = 4(a_1^2 + a_2^2)^{1/2} \) and
\[ f_i(x_i) = a_j(1 - |x_i/a_i|), \quad j \neq i. \]
Thus, from (5) and (6), we have
\[ \min(a_2/a_1, a_1/a_2) \leq p_2(a_1^2 + a_2^2)^{1/2} \leq \pi/2. \]
When \( a_1 = a_2 = S/\sqrt{2} \), the rhombus is a square of side \( S \), and (7) becomes \( \pi S \leq \pi/2 \), whereas the exact value to five places is \( p_2 S = 1.3765 \cdots \).

Next, consider the ellipse \( (x_i/a_i)^2 + (x_j/a_j)^2 \leq 1 \) for which
\[ f_i(x_i) = a_j(1 - (x_i/a_i)^2)^{1/2}, \quad j \neq i. \]
For simplicity, we worsen (6) by combining it with the classical isoperimetric inequality
\[ L^2 \geq 4\pi A, \]
where \( A \), the area of the ellipse, is \( \pi a_1 a_2 \). Thus, we have
\[ \left[ \max(a_1, a_2) \right]^{-1} \leq p_2 \leq (a_1 a_2)^{-1/2}. \]
For the special case of a disc when \( a_1 = a_2 = R \), we attain equality on both sides and \( p_2 R = 1 \).

REFERENCES

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