COMMUTING PROJECTIONS WITH ASSIGNED RANGES

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1. Let $\mathfrak{H}$ be a Hilbert space. A projection is a bounded idempotent linear operator. Orthogonal projections form a proper subclass of the class of all projections. A problem is to get a condition for there to exist commuting projections $E_1, \cdots, E_n$ with the same ranges as given orthogonal projections $P_1, \cdots, P_n$ respectively. This will be settled in terms of properties of the sublattice, generated by $P_1, \cdots, P_n$ in the lattice of all orthogonal projections. In case $n=2$, commuting projections with minimum norms are constructed. Another problem is to find commuting projections $E_1, \cdots, E_n$ in a Hilbert space $\mathfrak{K}$, containing $\mathfrak{H}$ as a subspace, such that $P_jx=PE_jx$ for $x\in\mathfrak{H}$, $j=1, 2, \cdots, n$, where P is the orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$. This will be proved to be always possible.

2. Let P and Q be orthogonal projections with ranges $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Then $P\wedge Q$ and $P\vee Q$ will denote the orthogonal projections onto $\mathfrak{M}\cap\mathfrak{N}$ and the closure of $\mathfrak{M}+\mathfrak{N}$ respectively. $\mathfrak{M}\ominus\mathfrak{N}$ will stand for the subspace $\mathfrak{M}\cap(\mathfrak{M}\cap\mathfrak{N})^{\perp}$ and $P\ominus Q$ is the orthogonal projection onto it.

If there exist commuting projections E and F with ranges $\mathfrak{M}$ and $\mathfrak{N}$ respectively, $\mathfrak{M}+\mathfrak{N}$ becomes the range of the projection $E+F-EF$, therefore it is a closed subspace.

If, conversely, $\mathfrak{M}+\mathfrak{N}$ is closed, in view of the well-known theorem of Kober (see [3]) the operators E' and F', which assign to each $x\in\mathfrak{M}+\mathfrak{N}$ the elements $u+v$ and $u+w$ respectively, are continuous, where u, v and w are uniquely determined by the relation $x=u+v+w$ with $u\in\mathfrak{M}\cap\mathfrak{N}$, $v\in\mathfrak{M}\ominus\mathfrak{N}$ and $w\in\mathfrak{N}\ominus\mathfrak{M}$. Then the operators $E=E'(P\vee Q)$ and $F=F'(P\vee Q)$ are commuting projections with ranges $\mathfrak{M}$ and $\mathfrak{N}$ respectively.

It was shown by Mackey [3, p. 166] that the closedness of $\mathfrak{M}+\mathfrak{N}$ is equivalent to the property that $(P\vee Q)\wedge Q'=(P\wedge Q')\vee Q$ for every orthogonal projection Q' with $Q\leqq Q'$.

Suppose again that there exist commuting projections E and F with ranges $\mathfrak{M}$ and $\mathfrak{N}$ respectively. Since the commutativity implies that $E\mathfrak{N}\subseteq\mathfrak{M}\cap\mathfrak{N}$ for any $x, y\in\mathfrak{H}$, $(P\ominus Q)x$ and EQy are orthogonal, therefore

$$\|(P\ominus Q)x\|^2 \leqq \|(P\ominus Q)x - EQy\|^2 \leqq \|E\|^2\cdot\|(P\ominus Q)x - Qy\|^2.$$

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Since, for fixed x, the minimum of $\|(P\ominus Q)x-Qy\|^2$, $y\in\mathfrak{H}$ is equal to $\|(P\ominus Q)x\|^2-\|Q(P\ominus Q)x\|^2$, it follows that

$$\|Q(P\ominus Q)x\|^2 \leqq \{1-\|E\|^{-2}\}\cdot\|(P\ominus Q)x\|^2,$$

consequently

$$\|(Q\ominus P)(P\ominus Q)\|^2 \leqq 1-\|E\|^{-2} < 1,$$

because

$$(Q\ominus P)(P\ominus Q) = Q(P\ominus Q).$$

Suppose, conversely, that $\|(Q\ominus P)(P\ominus Q)\|<1$. Then both $I-(Q\ominus P)(P\ominus Q)$ and $I-(P\ominus Q)(Q\ominus P)$ have bounded inverses and the operators

$$E_0 = P\wedge Q + (P\ominus Q)\{I-(Q\ominus P)(P\ominus Q)\}^{-1}\cdot(I-Q\ominus P)$$

and

$$F_0 = P\wedge Q + (Q\ominus P)\{I-(P\ominus Q)(Q\ominus P)\}^{-1}\cdot(I-P\ominus Q)$$

are commuting projections with ranges $\mathfrak{M}$ and $\mathfrak{N}$ respectively. This can be seen by using the expansion

$$\{I-(Q\ominus P)(P\ominus Q)\}^{-1} = \sum_{n=0}^{\infty}[(Q\ominus P)(P\ominus Q)]^n.$$

The definition shows that $E_0F_0=P\wedge Q$ and $E_0+F_0-E_0F_0=P\vee Q$ and that $E_0(I-F_0)$ and $F_0(I-E_0)$ have the same ranges as $P\ominus Q$ and $Q\ominus P$ respectively. Then it follows that

$$\begin{aligned}\|x\|^2-\|E_0F_0x\|^2 &\geqq \|(P\vee Q-P\wedge Q)x\|^2\\ &= \|E_0(I-F_0)x+F_0(I-E_0)x\|^2\\ &\geqq \{1-\|(Q\ominus P)(P\ominus Q)\|^2\}\cdot\|E_0(I-F_0)x\|^2,\end{aligned}$$

therefore

$$\|x\|^2 \geqq \{1-\|(Q\ominus P)(P\ominus Q)\|^2\}\|E_0x\|^2,$$

which, together with the inequality already obtained, implies that

$$\|E_0\| = \{1-\|(Q\ominus P)(P\ominus Q)\|^2\}^{-1/2}.$$

The norm of F_0 is shown to have the same value.

It is desired to give an expression for the quantity $\|(Q\ominus P)(P\ominus Q)\|$ in terms of $P\vee Q$ and $P\wedge Q$.

LEMMA. $\|(Q\ominus P)(P\ominus Q)\|=\|P\vee Q+P\wedge Q-P-Q\|$.

PROOF. Since

$$P\vee Q+P\wedge Q-P-Q=(P\ominus Q)\vee(Q\ominus P)-(P\ominus Q)-(Q\ominus P)$$

and $(P\ominus Q)\wedge(Q\ominus P)=0$, it suffices to prove $\|I-P-Q\|=\|QP\|$ under the assumption that $P\wedge Q=0$ and $P+Q=I$.

Since it is known [1, p. 70] that

$$\|I-P-Q\|=\operatorname{Max}\{\|QP\|,\|(I-Q)(I-P)\|\},$$

the assertion is true in case $\|QP\|=1$. If $\|QP\|<1$, in view of the foregoing result there exist the unique commuting projections E and F with the same ranges as P and Q respectively and their norms are equal to $\{1-\|QP\|^2\}^{-1}$. On the other hand, E^* and F^* are the unique commuting projections with the same ranges as $I-Q$ and $I-P$ respectively. The same discussion is applied to get

$$\|E\|=\|E^*\|=\{1-\|(I-P)(I-Q)\|^2\}^{1/2},$$

therefore

$$\|(I-P)(I-Q)\|=\|QP\|.$$

Use, finally, the relation

$$\|(I-P)(I-Q)\|=\|(I-Q)(I-P)\|.$$

In view of this lemma, the preceding results can be summarized in the following theorem.

THEOREM 1. *For a pair of orthogonal projections P, Q the following conditions are mutually equivalent*:

(1) *there exist commuting projections E and F with the same ranges as P and Q respectively*;

(2) $(P\vee Q)\wedge Q'=(P\wedge Q')\vee Q$ *for every orthogonal projection Q' with $Q\leqq Q'$*;

(3) $\|P\vee Q+P\wedge Q-P-Q\|<1$. *When one (and all) of these conditions is fulfilled, there exist such commuting projections E_0 and F_0 with minimum norm*:

$$\|E_0\|=\|F_0\|=\{1-\|P\vee Q+P\wedge Q-P-Q\|^2\}^{-1/2}.$$

A related problem is to ask when there exist doubly commuting projections E and F, i.e. $EF=FE$ and $E^*F=FE^*$, with the same ranges as P and Q respectively. The answer is, however, quite simple. It is the case (when and) only when $PQ=QP$. In fact, since the range of the projection $I-E^*$ is the orthogonal complement of the range of E and is invariant under F by the double commutativity, the range of E reduces F, i.e. $FP=PF$. A similar argument yields $PQ=QP$.

3. Let $P_1, \cdots, P_n$ be orthogonal projections with ranges $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ respectively. Consider the sublattice $L(P_1, \cdots, P_n)$, generated by $P_1, \cdots, P_n$ in the lattice of all orthogonal projections, and the sublattice $L(\mathfrak{M}_1, \cdots, \mathfrak{M}_n)$, generated by $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$ in the lattice of all (not necessarily closed) subspaces. If the lattice $L(\mathfrak{M}_1, \cdots, \mathfrak{M}_n)$ consists of closed subspaces only, it is isomorphic to the lattice $L(P_1, \cdots, P_n)$.

THEOREM 2. *For orthogonal projections $P_1, \cdots, P_n$ the following conditions are equivalent*:

(1) *there exist commuting projections $E_1, \cdots, E_n$ with the same ranges as $P_1, \cdots, P_n$ respectively*;

(2) *the sublattice $L(P_1, \cdots, P_n)$, generated by $P_1, \cdots, P_n$ in the lattice of all orthogonal projections, is distributive and possesses one (and both) of the following equivalent properties*:

(a) $\|P\vee Q+P\wedge Q-P-Q\|<1$ *for* $P, Q\in L(P_1, \cdots, P_n)$,

(b) $(P\vee Q)\wedge Q'=(P\wedge Q')\vee Q$ *for* P, $Q\in L(P_1, \cdots, P_n)$ *and for every orthogonal projection Q' with $Q\leqq Q'$.*

PROOF. Suppose that the condition (1) is fulfilled, then the lattice $L(\mathfrak{M}_1, \cdots, \mathfrak{M}_n)$ is distributive, where $\mathfrak{M}_j$ is the range of P_j, $j=1, 2, \cdots, n$. In fact, for any three members $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{N}_3$ in it there exist commuting projections F_1, F_2, F_3, which are polynomials of $E_1, \cdots, E_n$ and have ranges $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{N}_3$ respectively. Since $(\mathfrak{N}_1+\mathfrak{N}_2)\cap\mathfrak{N}_3$ becomes the range of the projection $F_1(I-F_2)F_3+F_2F_3$, it follows that

$$(\mathfrak{N}_1 + \mathfrak{N}_2) \cap \mathfrak{N}_3 = (\mathfrak{N}_1 \cap \mathfrak{N}_3) + (\mathfrak{N}_2 \cap \mathfrak{N}_3).$$

Further every member of $L(\mathfrak{M}_1, \cdots, \mathfrak{M}_n)$ is a closed subspace, because it is a range of a projection. Then in view of Theorem 1 and the comment preceding Theorem 2, the lattice $L(P_1, \cdots, P_n)$, which becomes isomorphic to the lattice $L(\mathfrak{M}_1, \cdots, \mathfrak{M}_n)$, is distributive and possesses the properties (a) and (b).

Suppose, conversely, that the condition (2) is fulfilled. Then as in the above discussion, the lattice $L(\mathfrak{M}_1, \cdots, \mathfrak{M}_n)$ is distributive and consists of closed subspaces only. Now let J_k(or L_k) denote generally a set, consisting of k integers in $\{1, 2, \cdots, n\}$. Put for k and J_k

$$\mathfrak{M}(J_k) = \bigcap_{j\in J_k} \mathfrak{M}_j, \qquad \mathfrak{M}^{(k)} = \sum_{\text{all } J_k} \mathfrak{M}(J_k)$$

and

$$\mathfrak{N}(J_k) = \mathfrak{M}(J_k) \ominus \mathfrak{M}^{(k+1)}, \qquad (\mathfrak{M}^{(n+1)} = \{0\}).$$

Let us prove by induction that

$$\mathfrak{M}_j = \sum_{k=1}^{n} \sum_{j \in J_k} \mathfrak{N}(J_k), \qquad j = 1, 2, \cdots, n.$$

This is trivial for $n=1$. Assuming that the assertion is generally true for $n-1$, consider the subspaces $\mathfrak{M}_j^* = \mathfrak{M}_j \cap \mathfrak{M}_n$, $j=1, 2, \cdots, n-1$ and define subspaces $\mathfrak{M}^*(J_k)$, $\mathfrak{M}^{*(k)}$ and $\mathfrak{N}^*(J_k)$ (with $J_k \subset \{1, 2, \cdots, n-1\}$) from $\mathfrak{M}_1^*, \cdots, \mathfrak{M}_{n-1}^*$ just as $\mathfrak{M}(J_k)$, $\mathfrak{M}^{(k)}$ and $\mathfrak{N}(J_k)$ were defined from $\mathfrak{M}_1, \cdots, \mathfrak{M}_n$. Then the distributivity shows that

$$\mathfrak{M}^{*(k)} = \sum_{J_k} [\mathfrak{M}_n \cap \mathfrak{M}(J_k)] = \mathfrak{M}_n \cap \mathfrak{M}^{(k+1)};$$

hence $\mathfrak{N}^*(J_k) = \mathfrak{N}(J_k, n)$. Since the property, in question, of $L(\mathfrak{M}_1, \cdots, \mathfrak{M}_n)$ implies that the lattice $L(\mathfrak{M}_1^*, \cdots, \mathfrak{M}_{n-1}^*)$ is distributive and consists of closed subspaces only, it follows from the induction assumption that

$$\mathfrak{M}_j^* = \sum_{k=1}^{n} \sum_{j \in J_k} \mathfrak{N}^*(J_k), \qquad j = 1, 2, \cdots, n-1.$$

But this can be written in the form

$$\mathfrak{M}_j \cap \mathfrak{M}_n = \sum_{k=2}^{n} \sum_{j, n \in J_k} \mathfrak{N}(J_k).$$

A similar argument proves that for any distinct i, j

$$\mathfrak{M}_j \cap \mathfrak{M}_i = \sum_{k=2}^{n} \sum_{i, j \in J_k} \mathfrak{N}(J_k).$$

Since for any j the distributivity implies that

$$\mathfrak{M}_j = \mathfrak{N}(j) + \mathfrak{M}_j \cap \mathfrak{M}^{(2)} = \mathfrak{N}(j) + \sum_{i \neq j} (\mathfrak{M}_j \cap \mathfrak{M}_i),$$

the above result yields the required expression for $\mathfrak{M}_j$.

Next let us prove linear independence of the family of all nontrivial $\mathfrak{N}(J_k)$'s. To this end, it suffices to prove that for any J_k

$$\mathfrak{N}(J_k) \cap \left\{ \sum_{L_k \neq J_k} \mathfrak{N}(L_k) + \mathfrak{M}^{(k+1)} \right\} = \{0\},$$

because $\mathfrak{N}(J_i) \subseteq \mathfrak{M}^{(k+1)}$ whenever $k < i$. The assertion results from the following relations, based on the distributivity

$$\mathfrak{N}(J_k) \cap \left\{ \sum_{L_k \neq J_k} \mathfrak{N}(L_k) + \mathfrak{M}^{(k+1)} \right\} \subseteq \mathfrak{M}(J_k)$$
$$\cap \left\{ \sum_{L_k \neq J_k} \mathfrak{M}(L_k) + \mathfrak{M}^{(k+1)} \right\} = \mathfrak{M}(J_k) \cap \mathfrak{M}^{(k+1)}.$$

Now since $\mathfrak{N}(J_k)$'s have the closed sum $\mathfrak{M}^{(1)}$, in view of Kober's theorem there exist mutually annihilating projections $E(J_k)$ with ranges $\mathfrak{N}(J_k)$ respectively. Then the operators

$$E_j = \sum_{k=1}^{n} \sum_{j \in J_k} E(J_k), \qquad j = 1, 2, \cdots, n,$$

are commuting projections with ranges $\mathfrak{M}_j$. This completes the proof.

COROLLARY. *Let each of two families of orthogonal projections* $\{P_1, \cdots, P_n\}$ *and* $\{Q_1, \cdots, Q_m\}$ *satisfy the condition in Theorem* 2. *If each* P_i *commutes with all* Q_j, *then the combined family* $\{P_1, \cdots, P_n, Q_1, \cdots, Q_m\}$ *satisfies the same condition, too.*

PROOF. With the same notations as in the proof of Theorem 2, each subspace in $L(\mathfrak{M}_1, \cdots, \mathfrak{M}_n)$ is invariant under all Q_j ,hence so is each $\mathfrak{N}(J_k)$, which leads to the commutativity of each E_i with all Q_j. Similar argument shows that there exist commuting projections $F_1, \cdots, F_m$ which have the same ranges as $Q_1, \cdots, Q_m$ respectively and commute with all E_i.

4. When $\mathfrak{H}$ is imbedded in a larger Hilbert space $\mathfrak{K}$, a linear operator S in $\mathfrak{K}$ is called a dilation of a linear operator T in $\mathfrak{H}$, in case $Sx = PTx$ for $x \in \mathfrak{H}$, where P is the orthogonal projection from $\mathfrak{K}$ onto $\mathfrak{H}$.

In a previous paper [2] we proved that if a pair of orthogonal projections admits dilations which are commuting orthogonal projections, then they necessarily commute with each other. In this connection the following theorem is of interest.

THEOREM 3. *Orthogonal projections* $P_1, \cdots, P_n$ *admit dilations* $E_1, \cdots, E_n$, *which are commuting projections.*

PROOF. Consider the orthogonal sum

$$\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{H}_n$$

where each $\mathfrak{H}_j$ is a copy of $\mathfrak{H}$. Imbed $\mathfrak{H}$ and $\mathfrak{H}_j$ canonically into $\mathfrak{K}$. The isomorphism from $\mathfrak{H}$ to $\mathfrak{H}_j$ will be denoted by I_j. Consider $n+1$ subspaces defined by

$$\mathfrak{N}_j = \{(P_j + I_jP_j)x;\, x \in \mathfrak{H}\}, \qquad j = 1, 2, \cdots, n$$

and

$$\mathfrak{N} = \left\{\left(I - \sum_{j=1}^{n} P_j - \sum_{j=1}^{n} I_jP_j\right)x;\, x \in \mathfrak{H}\right\}.$$

Then it is easy to see that these subspaces are linearly independent with a closed sum; therefore in view of Kober's theorem there exist mutually annihilating projections $E_1, \cdots, E_n$ with ranges $\mathfrak{N}_1, \cdots, \mathfrak{N}_n$ respectively. Since each $x \in \mathfrak{H}$ is written in the form

$$x = \sum_{j=1}^{n} (P_j + I_jP_j)x + \left(I - \sum_{j=1}^{n} P_j - \sum_{j=1}^{n} I_jP_j\right)x,$$

it follows that $E_jx = P_jx + I_jP_jx$, which means that E_j is a dilation of P_j.

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References

1. N. I. Akhiezer and I. M. Glazman, *Theory of linear operators in Hilbert space*, Ungar, New York, 1961.

2. T. Ando and M. Takaguchi, *Commuting dilations of self-adjoint operators*, Proc. Japan Acad. **41** (1965), 696–698.

3. G. Mackey, *On infinite-dimensional linear spaces*, Trans. Amer. Math. Soc. **57** (1945), 155–207.

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