THE LOCALIZATION OF THE STRICT TOPOLOGY VIA BOUNDED SETS[1]

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The β or strict topology on the space $C(S)$ of bounded continuous functions on a locally compact Hausdorff space S was introduced by Buck, see [1]. The purpose of this note is to prove:

THEOREM. *No properly stronger locally convex topology on $C(S)$ agrees with β on norm bounded sets (equivalently, on β-bounded sets).*

The most obvious consequence of this is that a linear transformation from $C(S)_\beta$ into a locally convex space is continuous if it is continuous on bounded sets, a fact which was observed for linear functionals by Buck in [1]. The β' or bounded strict topology is the strongest locally convex topology on $C(S)$ which agrees with β on bounded sets. For a proof of existence, see [2], [5], or [7], where an explicit neighborhood base is given. Our theorem may now be stated: $\beta' = \beta$. Thus, since β agrees with the compact open topology on norm bounded sets, we have that β is the strongest locally convex topology on $C(S)$ which agrees with the compact open topology on norm bounded sets. It is easy to show (see [5]), that $C(S)_{\beta'}$ has the same adjoint as $C(S)_\beta$, namely the functionals given by elements of $M(S)$, the space of bounded regular Borel measures on S. This immediately gives $\beta' = \beta$ if S is paracompact, as Conway showed in [4] that $C(S)_\beta$ is a Mackey space then. In [3], the β' topology is discussed in the context of general localizations (see [6, pp. 154 and 155] or [2, part 2]). In [5] and [7], the β' topology was useful in questions of continuity and equicontinuity of operators, and in neither paper was it apparent that β would have served as well.

PROOF OF THEOREM. Let $\| \quad \|$ denote the supremum norm on $C(S)$ and the variation norm on $M(S)$. Let $C_0(S)$ denote the space of functions in $C(S)$ which vanish at infinity, and for $\phi \in C_0(S)$, let

$$V_\phi = \{f \in C(S) : \|f\phi\| \leqq 1\}.$$

Then the sets $\{V_\phi\}$ form a neighborhood base at 0 for β. If $\psi \in C(S)$ and $\mu \in M(S)$, then $|\mu|$ denotes the variation (measure) of μ, and $\psi\mu \in M(S)$ is defined by $[\psi\mu](E) = \int_E \psi d\mu$.

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If $\psi \in C(s)$, $\psi \geqq 0$, then let

$$A_\psi = \{f \in C(S): |f| \leqq \psi\}.$$

We see that the polar of A_ψ is given by

$$(A_\psi)^0 = \{\mu \in M(S): \|\psi\mu\| \leqq 1\}.$$

Now let W be an absolutely convex β'-closed β' neighborhood of 0. For each $n = 1, 2, \cdots$, let B_n denote the closed norm ball in $C(S)$ of radius n, and let $\phi_n \in C_0(S)$ be such that $\phi_n \geqq 0$ and

$$W \cap B_n \supset B_n \cap V_{\phi_n}.$$

Let $\phi_n' = \max\{\phi_n, 1/n\}$, $\psi_n = 1/\phi_n'$, and $A_n = A_{\psi_n}$. Then

$$W \cap B_n \supset A_n = B_n \cap V_{\phi_n}.$$

Let W' denote the β'-closed absolutely convex hull of $\bigcup A_n$. Then $W' \subset W$, and $(W')^0 = \bigcap (A_n)^0$. We will show that $(W')^0$ is β-equicontinuous, so that W' is a β neighborhood of 0.

Since each $(A_n)^0$ is norm bounded, $(W')^0$ is norm bounded. Suppose $\epsilon > 0$. Let $(1/n) < \epsilon$, and

$$K = \{x \in S: |\phi_n(x)| \geqq 1/n\}.$$

If $\mu \in (W')^0$, then $\mu \in (A_n)^0$, so that

$$|\mu|(S\backslash K) = \int_{S\backslash K} d|\mu| = \int_{S\backslash K} \phi_n' \psi_n d|\mu| = \int_{S\backslash K} \phi_n' d|\psi_n\mu|$$
$$= (1/n) \int_{S\backslash K} d|\psi_n\mu| \leqq (1/n)\|\psi_n\mu\| < \epsilon.$$

Thus, $(W')^0$ is β-equicontinuous by Conway's characterization in [4].

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