PERMUTATIONS AND TWO SEQUENCES WITH THE SAME CLUSTER SET[1]

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Suppose that a $(n \to a_n)$ and b $(n \to b_n)$ are sequences in a compact metric space with distance d. Suppose further that a and b have the same set of cluster points C. Von Neumann [1, p. 11–12] proved there exists a permutation π of the set of positive integers Z such that $d(a_n, b_{\pi n}) \to 0$. Halmos [2] recently gave an improved proof (and the above statement for compact metric spaces). A discussion of this result may be found in [2]. My purpose is to give a shorter proof than that of Halmos, which used the Schröder-Bernstein Theorem.

PROOF. Let $\pi 1 = 1$. We now construct πn inductively for $n > 1$, given $\pi 1, \cdots, \pi(n-1)$. Write Z_1 for Z and Z_n for $Z - \{\pi 1, \cdots, \pi(n-1)\}$. Let $\rho(n) = \min Z_n$. Let πn be the smallest integer in Z_n such that

$$d(b_{\pi n}, a_n) \leqq d(a_n, C) + 1/\rho(n) + d(b_{\rho(n)}, C). \tag{1}$$

Such an element exists: Let c_n be chosen in the compact set C so that $d(a_n, c_n) = d(a_n, C)$. Then for some $m \in Z_n$, $d(b_m, c_n) \leqq 1/\rho(n)$, since c_n is a cluster point of b. Then $\pi n = m$ would satisfy (1). To see that π is "onto," i.e., $\{\pi 1, \pi 2, \cdots\} = Z$, let us suppose not. Let $q = \min Z - \{\pi 1, \pi 2, \cdots\}$. Then for all n, $\rho(n) \leqq q$, and $N \overset{\text{def}}{=} \max \{n : q \neq \rho(n)\}$ is finite. Choose $c^q \in C$ so that $d(b_q, c^q) = d(b_q, C)$. Since $c^q \in C$, c^q is a cluster point of a, so we may choose $n > N$ such that $1/q > d(a_n, c^q)$. Obviously, $d(a_n, C) + 1/q > d(a_n, c^q)$. Since $n > N$, $\rho(n) = q$, and since $\pi n > q$, $d(b_q, a_n) > d(a_n, C) + 1/q + d(b_q, c^q) > d(a_n, c^q) + d(b_q, c^q)$, contradicting the triangle inequality. Hence π is onto. Since $\pi n \to \infty$ and $d(a_n, C) \to 0$ and $d(b_{\pi n}, C) \to 0$, (1) implies $d(a_n, b_{\pi n}) \to 0$.

REFERENCES

1. J. von Neumann, *Charakterisierung des Spektrums eines Integraloperators*, Hermann, Paris, 1935.

2. P. R. Halmos, *Permutations of sequences and the Schröder-Bernstein theorem*, Proc. Amer. Math. Soc. **19** (1968), 509–510.

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Received by the editors August 14, 1968.

[1] Partially supported by National Science Foundation Grant GP-7846.