THE STONE-ČECH COMPACTIFICATION OF AN IRREDUCIBLY CONNECTED SPACE

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1. **Introduction.** A connected topological space X is irreducibly connected about a subset $A \subset X$ (written X is an A-i-connex) if no proper connected subspace of X contains A. A connected space Y is irreducibly closed connected about a subset $B \subset Y$ (written Y is a B-i-C-connex) if no proper closed connected subspace of Y contains B. The structure of such spaces has been studied by Gehman [**1**], Wilder [**7**] and Strebe [**3**], [**4**], among others. In this paper we show that the Stone-Čech compactification βX of an i-connex X is an i-connex, and that βY, for Y a suitably restricted i-C-connex, is an i-C-connex.

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2. **Notation.** All spaces are at least T_1 and completely regular. For $S \subset X$, Cl S is the closure of S in X, and for $T \subset \beta X$, $\mathrm{Cl}_\beta\, T$ is the closure of T in βX. If $S \subset X$, then $S^0 = \beta X - \mathrm{Cl}_\beta\,(X - S)$.

3. ***i*-connexes.**

THEOREM 1. *If X is an A-i-connex, then βX is an $A \cup (\beta X - X)$-i-connex.*

PROOF. Since X is connected, each point of $\beta X - X$ is a noncut point of βX. Thus any set about which βX is an i-connex must contain $\beta X - X$ [**1**, Theorem 3, p. 545].

If there is a subspace $X' \neq \beta X$ with $A \cup (\beta X - X) \subset X'$, then for any $x \in \beta X - X'$, $x \in X - A$ so x is a cut point of X. However, any cut point of X is a cut point of βX because an open subset U of βX is connected if and only if $U \cap X$ is connected [**2**, Lemma 1.4, p. 575]. Thus for $x \in \beta X - X'$, $\beta X - \{x\} = P \cup Q$ (sep).

If X' is connected, then $X' \subset P$, say. Now $Q \cup \{x\}$, being a continuum, has a noncut point $z \neq x$ [**5**, Theorem 1.11, p. 491]. Further $z \in X - A$. It follows that

$$\beta X - \{z\} = (P \cup \{x\}) \cup [(Q \cup \{x\}) - \{z\}]$$

is connected; hence that $X - \{z\}$ is a connected proper subspace of X containing A, a contradiction.

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To see that no converse is possible, let X be the nonnegative reals with the usual topology and $A = \{0\}$.

4. i-C-connexes.

THEOREM 2. *If the A-i-C-connex X is normal and is semilocally connected at each $x \in X - A$, then βX is an A-i-C-connex.*

PROOF. If there is a proper closed connected subspace K of βX with $A \subset K$, then for $x \in X \cap (\beta X - K)$ there is an X-open neighborhood V of x with $\mathrm{Cl}_\beta V \cap K = \emptyset$. Since X is semilocally connected at x, there is an X-open set U with $x \in U \subset V$ such that $X - U$ has finitely many components $C_1, \cdots, C_n$. Further, $n \geqq 2$ and $A \cap C_i \neq \emptyset$ for at least two indices i.

Now X is normal and therefore the closures in βX of $C_1, \cdots, C_n$ are mutually disjoint closed connected sets [6, Theorem 1, p. 97], i.e., they are the components of $\mathrm{Cl}_\beta(X - U)$. It follows that $A \subset K \subset \mathrm{Cl}_\beta C_j$ for some index j, a contradiction.

THEOREM 3. *If the A-i-C-connex X is normal and locally connected, then βX is an A-i-C-connex.*

PROOF. If there is a proper closed connected subspace K of βX with $A \subset K$, then there is a connected X-open set U with $\mathrm{Cl}_\beta U \cap K = \emptyset$. $X - U$ is not connected, so from $X - U = \mathrm{Cl}(X - \mathrm{Cl}\ U)$ it follows that $X - \mathrm{Cl}\ U$ is not connected.

If C is a component of $X - \mathrm{Cl}\ U$, then C is open in X and $X - C$ is closed and connected. Therefore $C \cap A \neq \emptyset$ for each component C of $X - \mathrm{Cl}\ U$. Thus $X - \mathrm{Cl}\ U = P \cup Q$(sep) with $P \cap A \neq \emptyset$, $Q \cap A \neq \emptyset$.

Now $K \subset \beta X - \mathrm{Cl}_\beta U = (X - \mathrm{Cl}\ U)^0 = (P \cup Q)^0$, and by [6, Lemma 2, p. 98] $(P \cup Q)^0 = P^0 \cup Q^0$ and $(P \cap Q)^0 = P^0 \cap Q^0$. Since K is connected, $K \subset P^0$, say; however, $A \cap Q^0 \supset A \cap Q \neq \emptyset$ and this is a contradiction.

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