

## MONOTONE $n$ -FRAMES ARE TAME<sup>1</sup>

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An  $n$ -frame  $\mathfrak{F}_n$  is the union of  $n$  arcs which are disjoint except for a common end point called the *branch point* of  $\mathfrak{F}_n$ . An  $n$ -frame of  $E^3$  is *tamely imbedded* provided there is an autohomeomorphism of  $E^3$  which carries it onto a polygonal  $n$ -frame. We will say that an  $n$ -frame  $\mathfrak{F}_n$  in  $E^3$  is *monotone* provided each geometric 2-sphere centered at the branch point of  $\mathfrak{F}_n$  intersects each of the  $n$  defining arcs of  $\mathfrak{F}_n$  in at most one point. Because each monotone  $n$ -frame is of the same imbedding type as a monotone  $n$ -frame having its branch point at the origin and its end points on the unit sphere, we will assume that a monotone  $n$ -frame has these properties as well as a prescribed ordering, say  $a_1, a_2, \dots, a_n$ , of its defining arcs.

Since for each  $s \in (0, 1]$ , the arc  $a_i$  intersects the sphere of radius  $s$  centered at the origin in a single point  $p_i(s)$ , the equation  $\phi(s) = (p_1(s), \dots, p_n(s))$ ,  $0 < s \leq 1$ , defines a continuous function from  $(0, 1]$  into the space  $F_n(S^2)$  of all  $n$ -tuples of distinct points of  $S^2$ . In this way there is established a one-one correspondence between monotone  $n$ -frames  $\mathfrak{F}_n$  and continuous maps  $\phi$  of  $(0, 1]$  into  $F_n(S^2)$ . We restrict our attention to frames  $\mathfrak{F}_n$  for which the  $n$ -tuple of end points  $(p_1(1), \dots, p_n(1))$  is the base point  $(p_1, \dots, p_n)$  of  $F_n(S^2)$ .

Let  $LH(B^3)$  be the space of all orientation preserving autohomeomorphisms  $H$  of the unit ball  $B^3$  which are level preserving in the sense that  $\|H(x)\| = \|x\|$  for all  $x \in B^3$ . To each  $H \in LH(B^3)$  there corresponds a family  $h_s$  of autohomeomorphisms of the boundary sphere  $S^2$  of  $B^3$  defined by  $h_s(x) = H(sx)/s$  for  $s \in (0, 1]$ ,  $x \in S^2$ . Therefore  $LH(B^3)$  is in one-one correspondence with the set of maps of  $(0, 1]$  into the group  $G(S^2)$  of all orientation preserving autohomeomorphisms of  $S^2$ .

In order to relate monotone  $n$ -frames  $\mathfrak{F}_n$  and level preserving homeomorphisms  $H$ , consider the map  $\rho_n: G(S^2) \rightarrow F_n(S^2)$  defined by  $\rho_n(f) = (f(p_1), \dots, f(p_n))$ , where  $(p_1, \dots, p_n)$  is the base point of  $F_n(S^2)$ .

The following theorem is patterned after a result of G. S. McCarty [3, Lemma 4.1], but, as used here, is better viewed as a generalization of a fundamental theorem of E. Artin in braid theory [1, p. 104]. (See

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Presented to the Society, August 28, 1968; received by the editors July 3, 1967.

<sup>1</sup> Prepared with partial support from the Office of Scientific and Scholarly Research of the Graduate School of the University of Oregon.

D. M. Dahm [2, p. 15], where this generalization is developed for braids on arbitrary manifolds.) Although the theorem is stated for  $S^2$ , the proof given is valid for any connected manifold of dimension at least 2.

**THEOREM 1.**  $G(S^2)$  is a fiber bundle over  $F_n(S^2)$  relative to the map  $\rho_n$ .

**PROOF.**  $G(S^2)$  is an effective transitive group of transformations of  $F_n(S^2)$ . Consequently, by the bundle structure theorem, it suffices to show that  $\rho_n$  admits a local cross-section. But, since the basic open sets in  $F_n(S^2)$  may be chosen to be products of  $n$  disjoint open discs, this is a consequence of the following elementary fact. Let  $G(B^2, S^1)$  be the group of autohomeomorphisms of the 2-cell  $B^2$  which are point-wise fixed on the boundary  $S^1$ , and let  $\pi$  be the map of  $G(B^2, S^1)$  onto the interior of  $B^2$  defined by  $\pi(f) = f(0)$ . Then  $\pi$  admits a global cross section.

As a corollary to Theorem 1, we have our main result.

**THEOREM 2.** Any monotone  $n$ -frame is tame.

**PROOF.** Let  $\mathcal{F}_n$  be a monotone  $n$ -frame. Then  $\mathcal{F}_n$  corresponds to a map  $\phi: (0, 1] \rightarrow F_n(S^2)$ . Choosing the base point  $(p_1, \dots, p_n)$  to be  $\phi(1)$ , the map  $\phi$  lifts to a map  $h: (0, 1] \rightarrow G(S^2)$  such that  $h_1$  is the identity. The corresponding map  $H$  in  $LH(B^3)$  is an autohomeomorphism of  $B^3$  carrying  $\mathcal{F}_n$  onto  $n$  radial segments in  $B^3$ . Since  $H$  is the identity on the boundary sphere,  $H$  extends to all of  $E^3$  and hence  $\mathcal{F}_n$  is tame.

**ADDED IN PROOF.** The authors are grateful to Professor R. H. Bing for pointing out that Theorem 2 (generalized, as noted, to  $n$ -manifolds) is equivalent to Proposition 2.3 of: R. H. Bing and V. L. Klee, *Every simple closed curve in  $E^3$  is unknotted in  $E^4$* , J. London Math. Soc. **39** (1964), 86-94.

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