ON FINITE DECOMPOSITIONS OF $E^{2n-1}$

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I. Introduction and results. For any upper monotone decomposition $G$ of $E^m$, with only finitely many nondegenerate elements, it is known, [5], that the decomposition space $E^m/G$ of $G$ is embeddable in $E^{m+2}$.

For $m = 3$, the dimension $m + 2 = 5$ is the best possible one, in the sense that there are monotone decompositions $G$ of $E^3$, with finitely many nondegenerate elements, for which $E^3/G$ is not embeddable in $E^4$; see [1], [3] and [6].

The purpose of this paper is to generalize the last result for all odd $m$.

All the complexes in this paper are to be understood as finite geometrical ones in some Euclidean space. For two complexes $K$ and $L$, $K \vee L$ will denote their join, (provided they are situated properly).

$C^q_p$ will denote the $p$-skeleton of the $(q - 1)$-simplex.

Our result is the following

**Theorem.** For each integer $n$, $n > 1$, there exists a monotone decomposition $G$ of $E^{2n-1}$, with only $2n + 3$ nondegenerate elements, such that $E^{2n-1}/G$ is not embeddable in $E^{2n}$. The nondegenerate elements of $G$ may be chosen so as to be (p.w.l.) homeomorphic to $C^{2n+2}_{2n+2}$.

With very little effort we can simplify the nondegenerate elements of such a decomposition $G$, while increasing the number of them, as follows:

**Corollary.** For each integer $n$, $n > 1$, there exists a monotone decomposition $G$ of $E^{2n-1}$, with only $3n + 3$ nondegenerate elements, such that $E^{2n-1}/G$ is not embeddable in $E^{2n}$; where the nondegenerate elements may be so chosen as to be (p.w.l.) homeomorphic to $C^3_2 \vee \cdots \vee C^3_n$ (n times).

As can be easily shown, the theorem is valid even for $n = 1$, where $G$ is only u.s.c. For example, if the nondegenerate elements of an u.s.c. decomposition $G$ of $E^1$ are the following sets of points $\{1; 4; 11\}$, $\{2; 9\}$, $\{3; 6\}$, $\{5; 8\}$ and $\{7; 10\}$ then the subspace $[1, 11]/G$ of $E^1/G$ is homeomorphic to the nonplanar graph $C^1_6$.

For $2n - 1 = 3$, the results of [1], [3] and [6] are better than ours,
however, to the best knowledge of the author of this paper, no other similar results are known for \( m > 3 \), \((2n - 1 > 3)\).

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II. Proof of the Theorem. Let \( n > 1 \) be given, and let \( P_1, \ldots, P_{2n+3} \) be the vertices of \( C_{2n+3}^n \), where \( C_{2n+3}^n \) is in some \( E^d \).

Let \( \gamma^2C_{2n+3}^n \) denote the second barycentric subdivision of \( C_{2n+3}^n \) and let

\[
1) \quad K_n = C_{2n+3}^n - \text{open star } (\cup_{i=1}^{2n+3} P_i, \gamma^2C_{2n+3}^n) \quad \text{and, for each} \quad 1 \leq i \leq 2n+3, \quad \text{let} \quad T_i^n = \text{link } (P_i, \gamma^2C_{2n+3}^n).
\]

Therefore \( T_i^n \subset \text{Bd} \ K_n \), for all \( 1 \leq i \leq 2n+3 \), and \( i \neq j \) implies \( T_i^n \cap T_j^n = \emptyset \).

Let \( \sigma \) be any \( q \)-simplex of \( C_{2n+3}^n \), and let \( \tilde{\sigma} \) be \( \sigma \cap K_n \), then \( \tilde{\sigma} \) is a \( q \)-cell, \( q \geq 1 \).

\( \tilde{\sigma} \cap T_i^n \) is a \((q-1)\)-simplex of \( T_i^n \) on the boundary of \( \tilde{\sigma} \), or it is empty, according to whether \( P_i \subset \sigma \) or not.

Therefore, the effect on \( \tilde{\sigma} \) of shrinking all the \( T_i^n - \sigma \) of \( K_n \), each one to a point, will be that of shrinking \( q + 1 \) mutually disjoint \((q-1)\)-simplexes on the boundary of \( \tilde{\sigma} \), each one to a point, the result of which is again a \( q \)-cell.

Moreover, if \( \sigma_1, \ldots, \sigma_t \) are all the simplexes of \( C_{2n+3}^n \) which contain one of the \( P_i - \sigma \), say \( P_{j_0} \), then \( \tilde{\sigma}_i \cap T_{j_0}^n \neq \emptyset \) for all \( 1 \leq i \leq t \), and therefore, after shrinking all the \( T_i^n - \sigma \) of \( K_n \), each one to a point, the resulting cells, which corresponds to \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_t \), will have that point, which corresponds to \( T_{j_0}^n \), in common. Therefore

\[
2) \quad \text{The resulting set, after shrinking all the } T_i^n - \sigma \text{ of } K_n, \text{ each one to a point, is homeomorphic to } C_{2n+3}^n.
\]

Now, \( C_{2n+3}^n \) is not embeddable in \( E^{2n} \), by [2]; each \( T_i^n \) is (p.w.l.) homeomorphic to \( C_{2n+3}^{n-1} \) and there are \( 2n+3 \) \( T_i^n - \sigma \), therefore we have just shown, that our theorem will follow, provided we can show that

\[
3) \quad K_n \text{ is (p.w.l.) embeddable in } E^{2n-1}.
\]

To do this, we begin with an embedding of the \( n - 1 \) skeleton, \( C_{2n+3}^{n-1} \), of \( C_{2n+3}^n \), geometrically and in general position in \( E^{2n-1} \), as follows:

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* It was suggested by the referee that this can be done as follows: \( K_n \) collapses by Whitehead elementary collapsing to an \((n-1)\)-complex \( L_{n-1} \), which may be taken as the union of the \( n - 1 \) skeletons of the links of the vertices of a \((2n+2)\)-simplex, taken in the first barycentric subdivision. So one can embed \( L_{n-1} \) in \( E^{2n-1} \) and then undo the collapsing by elementary expansions, piping the 0-dimensional singularities off free edges as needed.

We prefer to present our proof, which is a simpler, though a longer, one.
For \( t_0 < t_1 < \cdots < t_{2n+3} \), let \( P(t_i) = (t_i, t_i^2, \cdots, t_i^{2n-1}) \in E^{2n-1} \), taking \( P_i \) to \( P(t_i) \), for each \( 1 \leq i \leq 2n+3 \), can be extended to the required embedding of \( C^{2n+3}_{2n+3} \) in \( E^{2n-1} \), using Theorem 1 of [4, p. 62]. For simplicity, we call \( P(t_i) \) by \( P_i \), for each \( 1 \leq i \leq 2n+3 \).

From the general position of the \( P_i - s \) it follows, that the affine hulls of each \( n+1 \) points among the \( P_i - s \) are different from each other, since, otherwise, there will be some \( k+2 \) points on a \( k \)-dimensional affine flat, for \( k \leq n \).

Adding one point at a time, in general position with respect to the previously such points, let \( A_{ij} \) be a point near \( P_i \) on the segment \( P_i P_j \), for all \( i, j \in \{1, \cdots, 2n+3\} \), with \( n+1 \) elements, and for each \( k \in \mathbb{N} \), we define \( S_k^N = \text{convex} \{ A_{ik} \mid i \in \mathbb{N}, i \neq k \} \).

From the general position of the \( A_{ij} - s \), it follows that (relative interior of \( S_k^N \)) \( \cap \) (relative interior of \( S_{k'}^N \)) = \( \emptyset \), provided either \( N - \{k\} \neq N' - \{k'\} \), or else \( N - \{k\} = N' - \{k'\} \) but then \( k \neq k' \).

Denoting the cone over a set \( A \) with vertex at \( V \) by \( \text{cone}_V \), we define \( T_{n-1} = [C^{2n+3}_{2n+3} - U_{n; k \in \mathbb{N}} \text{cone} P_k \text{ (relative boundary } S_k^N)] \cup \bigcup_{N; k \in \mathbb{N}} S_k^N \).

Each \( k \) simplex of \( C^{2n+3}_{2n+3} \) "loses" a regular neighborhood of each one of its vertices, in \( T_{n-1} \).

For each \( N = \{j_1, \cdots, j_{n+1}\} \) as above, we define \( R_N \) by: \( R_N = \bigcup_{i=1}^{n+1} \text{convex} \{ P_{j_i}, \cdots, \hat{P}_{j_i}, \cdots, P_{j_{n+1}} \} \). \( R_N \) is the relative boundary of the \( n \)-simplex \( \{P_{j_1}, \cdots, P_{j_{n+1}}\} \) of \( C^{2n+3}_{2n+3} \), which is "replaced" in \( T_n \) by \( R_N' \), where \( R_N' = [R_N - U_{i=1}^{n+1} \text{cone} P_{j_i} \text{ (relative boundary } S_k^N)] \cup \bigcup_{i=1}^{n+1} S_i^{N'} \).

\( R_N' \) is the relative boundary of a convex polytope \( B_N \). Let \( D_N \) be a point in the relative interior of \( B_N \), such that \( N \neq N' \) implies \( D_N \neq D_{N'} \), which is possible since \( B_N \subset \text{affine hull} \ R_N' = \text{affine hull} \ R_N = \text{affine hull} \ \{P_i \mid i \in \mathbb{N}\} \), and the \( P_i - s \) are chosen to be in general position.

Let us choose, for each \( N \) as above, an index \( i_N \), where \( i_N \in \mathbb{N} \), and let us define \( R_N'' = \text{closure} (R_N' - S^{N'}_{N}) \). \( R_N'' \) is an \((n-1)\) polyhedral cell.

We add, one at a time, a collar to \( R_N'' \) in the direction towards \( D_N \), —of the form

\[
\{ t D_N + (1 - t) x \mid x \in R_N'' \text{ and } 0 \leq t \leq \epsilon_N \},
\]

where \( \epsilon_N > 0 \) is so small as to avoid any intersection, of this collar with all the previous resulting set, except at \( R_N'' \). This is possible because: (a) \( D_N \subset \text{relative interior of } B_N \), and the additions of each new collar are performed in distinct affine flats, and (b) the collars are to be so small as to avoid meeting the compact disjoint set.
The result is an \( n \)-dimensional polyhedral set, in \( E^{2n-1} \), which we claim, is (p.w.l.) homeomorphic to \( K_n \), as defined in (1). The added collar to \( R_n' \) corresponds, in \( K_n \), to the set
\[
\text{closure} \left[ (\text{convex} \{ P_i \mid i \in N \}) - \text{open star} \left( \bigcup_{i=1}^{2n+3} P_i, \gamma^2 C_{2n+3}^n \right) \right].
\]

If we let \( S_N^* \) denote the closure of (the relative boundary of the collar added to \( R_n' \), less the set \( R_n' \)), then what corresponds in our resulting set to \( T_n^* \) of \( K_n \) is
\[
\bigcup_{N \text{ if } \gamma \neq 4} S_N^* \bigcup_{N \text{ if } \gamma = 4} S_N^*.
\]

Therefore, we have just proved (3), hence the proof of the theorem is completed.

Figures 1, 2 describe the constructions, for \( n = 2 \), concerning only three of the \( R_n - s \):

The proof of the corollary is similar to the proof of the theorem

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Figure 1}
\end{figure}
but where we deal with \( C^0_3 \cup \cdots \cup C^0_3 \) \((n+1\) times) instead of \( C^n_{2n+3} \), and make the obvious corrections for the notations. A similar result can be obtained in the same way, involving any known \( n \)-complex which is not embeddable in \( E^{2n} \).

**References**


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