ON FINITE DECOMPOSITIONS OF E^{2n-1}

JOSEPH ZAKS[1]

I. **Introduction and results.** For any upper monotone decomposition G of E^m, with only finitely many nondegenerate elements, it is known, [5], that the decomposition space E^m/G of G is embeddable in E^{m+2}.

For $m=3$, the dimension $m+2=5$ is the best possible one, in the sense that there are monotone decompositions G of E^3, with finitely many nondegenerate elements, for which E^3/G is not embeddable in E^4; see [1], [3] and [6].

The purpose of this paper is to generalize the last result for all odd m.

All the complexes in this paper are to be understood as finite geometrical ones in some Euclidean space. For two complexes K and L, $K \vee L$ will denote their join, (provided they are situated properly).

C_q^p will denote the p-skeleton of the $(q-1)$-simplex.

Our result is the following

THEOREM. *For each integer n, $n>1$, there exists a monotone decomposition G of E^{2n-1}, with only $2n+3$ nondegenerate elements, such that E^{2n-1}/G is not embeddable in E^{2n}. The nondegenerate elements of G may be chosen so as to be (p.w.l.) homeomorphic to C_{2n+2}^{n-1}.*

With very little effort we can simplify the nondegenerate elements of such a decomposition G, while increasing the number of them, as follows:

COROLLARY. *For each integer n, $n>1$, there exists a monotone decomposition G of E^{2n-1}, with only $3n+3$ nondegenerate elements, such that E^{2n-1}/G is not embeddable in E^{2n}; where the nondegenerate elements may be so chosen as to be (p.w.l.) homeomorphic to $C_3^0 \vee \cdots \vee C_3^0$ (n times).*

As can be easily shown, the theorem is valid even for $n=1$, where G is only u.s.c. For example, if the nondegenerate elements of an u.s.c. decomposition G of E^1 are the following sets of points $\{1;4;11\}$, $\{2;9\}$, $\{3;6\}$, $\{5;8\}$ and $\{7;10\}$ then the subspace $[1, 11]/G$ of E^1/G is homeomorphic to the nonplanar graph C_5^1.

For $2n-1=3$, the results of [1], [3] and [6] are better than ours,

Presented to the Society, June 15, 1968; received by the editors February 26, 1968.
[1] Research supported in part by the National Science Foundation Grant GP-7035.

however, to the best knowledge of the author of this paper, no other similar results are known for $m>3$, $(2n-1>3)$.

The author of this paper would like to express his highly appreciated acknowledgement to Professor Branko Grünbaum, whose help was essential in simplifying this paper.

II. **Proof of the Theorem.** Let $n>1$ be given, and let $P_1, \cdots, P_{2n+3}$ be the vertices of C^n_{2n+3}, where C^n_{2n+3} is in some E^d.

Let $\gamma^2 C^n_{2n+3}$ denote the second barycentric subdivision of C^n_{2n+3}, and let

(1) $K_n = C^n_{2n+3} - \text{open star } (\bigcup_{i=1}^{2n+3} P_i, \quad \gamma^2 C^n_{2n+3})$ and, for each $1 \leqq i \leqq 2n+3$, let $T^n_i = \text{link } (P_i, \gamma^2 C^n_{2n+3})$.

Therefore $T^n_i \subset \text{Bd } K_n$, for all $1 \leqq i \leqq 2n+3$, and $i \neq j$ implies $T^n_i \cap T^n_j = \emptyset$.

Let σ be any q-simplex of C^n_{2n+3}, and let $\tilde{\sigma}$ be $\sigma \cap K_n$, then $\tilde{\sigma}$ is a q-cell, $q \geqq 1$.

$\tilde{\sigma} \cap T^n_i$ is a $(q-1)$-simplex of T^n_i on the boundary of $\tilde{\sigma}$, or it is empty, according to whether $P_i \in \sigma$ or not.

Therefore, the effect on $\tilde{\sigma}$ of shrinking all the $T^n_i - s$ of K_n, each one to a point, will be that of shrinking $q+1$ mutually disjoint $(q-1)$-simplexes on the boundary of $\tilde{\sigma}$, each one to a point, the result of which is again a q-cell.

Moreover, if $\sigma_1, \cdots, \sigma_t$ are all the simplexes of C^n_{2n+3} which contain one of the $P_i - s$, say P_{j_0}, then $\tilde{\sigma}_i \cap T^n_{j_0} \neq \emptyset$ for all $1 \leqq i \leqq t$, and therefore, after shrinking all the $T^n_i - s$ of K_n, each one to a point, the resulting cells, which corresponds to $\tilde{\sigma}_1, \cdots, \tilde{\sigma}_t$, will have that point, which corresponds to $T^n_{j_0}$, in common. Therefore

(2) The resulting set, after shrinking all the $T^n_i - s$ of K_n, each one to a point, is homeomorphic to C^n_{2n+3}.

Now, C^n_{2n+3} is not embeddable in E^{2n}, by [2]; each T^n_i is (p.w.l.) homeomorphic to C^{n-1}_{2n+2} and there are $2n+3$ $T_i - s$, therefore we have just shown, that our theorem will follow, provided we can show that

(3) K_n is (p.w.l.) embeddable in E^{2n-1}.[2]

To do this, we begin with an embedding of the $n-1$ skeleton, C^{n-1}_{2n+3}, of C^n_{2n+3}, geometrically and in general position in E^{2n-1}, as follows:

---

[2] It was suggested by the referee that this can be done as follows: K_n collapses by Whitehead elementary collapsing to an $(n-1)$-complex L_{n-1}, which may be taken as the union of the $n-1$ skeletons of the links of the vertices of a $(2n+2)$-simplex, taken in the first barycentric subdivision. So one can embed L_{n-1} in E^{2n-1} and then undo the collapsing by elementary expansions, piping the 0-dimensional singularities off free edges as needed.

We prefer to present our proof, which is a simpler, though a longer, one.

For $t_1 < t_2 < \cdots < t_{2n+3}$, let $P(t_i) = (t_i, t_i^2, \cdots, t_i^{2n-1}) \in E^{2n-1}$, taking P_i to $P(t_i)$, for each $1 \leqq i \leqq 2n+3$, can be extended to the required embedding of C_{2n+3}^{n-1} in E^{2n-1}, using Theorem 1 of [4, p. 62]. For simplicity, we call $P(t_i)$ by P_i, for each $1 \leqq i \leqq 2n+3$.

From the general position of the $P_i - s$ it follows, that the affine hulls of each $n+1$ points among the $P_i - s$ are different from each other, since, otherwise, there will be some $k+2$ points on a k-dimensional affine flat, for $k \leqq n$.

Adding one point at a time, in general position with respect to the previously such points, let A_{ij} be a point near P_i on the segment P_iP_j, for all $i \neq j$, $1 \leqq i, j \leqq 2n+3$.

For each subset N of $\{1, \cdots, 2n+3\}$, with $n+1$ elements, and for each $k \in N$, we define $S_N^k = \text{convex } \{A_{ki} \mid i \in N, i \neq k\}$.

From the general position of the $A_{ij} - s$, it follows that (relative interior of S_N^k)$\cap$(relative interior of $S_{N'}^{k'}$)$= \emptyset$, provided either $N - \{k\} \neq N' - \{k'\}$, or else $N - \{k\} = N' - \{k'\}$ but then $k \neq k'$.

Denoting the cone over a set A with vertex at V by $\text{cone}_V A$, we define $T_{n-1} = [C_{2n+3}^{n-1} - \bigcup_{N;k \in N} \text{cone}_{P_k}(\text{relative boundary } S_N^k)] \cup \bigcup_{N;k \in N} S_N^k$.

Each k simplex of C_{2n+3}^{n-1} "loses" a regular neighborhood of each one of its vertices, in T_{n-1}.

For each $N = \{j_1, \cdots, j_{n+1}\}$ as above, we define R_N by: $R_N = \bigcup_{i=1}^{n+1} \text{convex } \{P_{j_1}, \cdots, \hat{P}_{j_i}, \cdots, P_{j_{n+1}}\}$. R_N is the relative boundary of the n-simplex $\{P_{j_1}, \cdots, P_{j_{n+1}}\}$ of C_{2n+3}^{n} in C_{2n+3}^{n-1}, which is "replaced" in T_n by R_N', where $R_N' = [R_N - \bigcup_{i=1}^{n+1} \text{cone}_{P_{j_i}} (\text{relative boundary } S_N^{j_i})] \cup \bigcup_{i=1}^{n+1} S_N^{j_i}$.

R_N' is the relative boundary of a convex polytope B_N. Let D_N be a point in the relative interior of B_N, such that $N \neq N'$ implies $D_N \neq D_{N'}$, which is possible since $B_N \subset \text{affine hull } R_N' = \text{affine hull } R_N = \text{affine hull } \{P_i \mid i \in N\}$, and the $P_i - s$ are chosen to be in general position.

Let us choose, for each N as above, an index i_N, where $i_N \in N$, and let us define $R_N'' = \text{closure } (R_N' - S_N^{i_N})$. R_N'' is an $(n-1)$ polyhedral cell.

We add, one at a time, a collar to R_N'' in the direction towards D_N, —of the form

$$\{tD_N + (1-t)x \mid x \in R_N'' \quad \text{and} \quad 0 \leqq t \leqq \epsilon_N\},$$

where $\epsilon_N > 0$ is so small as to avoid any intersection, of this collar with all the previous resulting set, except at R_N''. This is possible because: (a) $D_N \in$ relative interior of B_N, and the additions of each new collar are performed in distinct affine flats, and (b) the collars are to be so small as to avoid meeting the compact disjoint set

$$(\text{relative interior } B_n) \cap \left[\bigcup_N R''_N \cup \text{all previously added collars}) \right].$$

The result is an n-dimensional polyhedral set, in E^{2n-1}, which, we claim, is (p.w.l.) homeomorphic to K_n, as defined in (1). The added collar to R''_N corresponds, in K_n, to the set

$$\text{closure}\left[(\text{convex}\{P_i \mid i \in N\}) - \text{open star}\left(\bigcup_{i=1}^{2n+3} P_i, \gamma^2 C^n_{2n+3} \right) \right].$$

If we let S^*_N denote the closure of (the relative boundary of the collar added to R''_N, less the set R''_N), then what corresponds in our resulting set to T^n_i of K_n is

$$\bigcup_{N \text{ if } i_N \neq i} S^i_N \cup \bigcup_{N \text{ if } i_N = i} S^*_N.$$

Therefore, we have just proved (3), hence the proof of the theorem is completed.

Figures 1, 2 describe the constructions, for $n=2$, concerning only three of the $R_N - s$:

The proof of the corollary is similar to the proof of the theorem

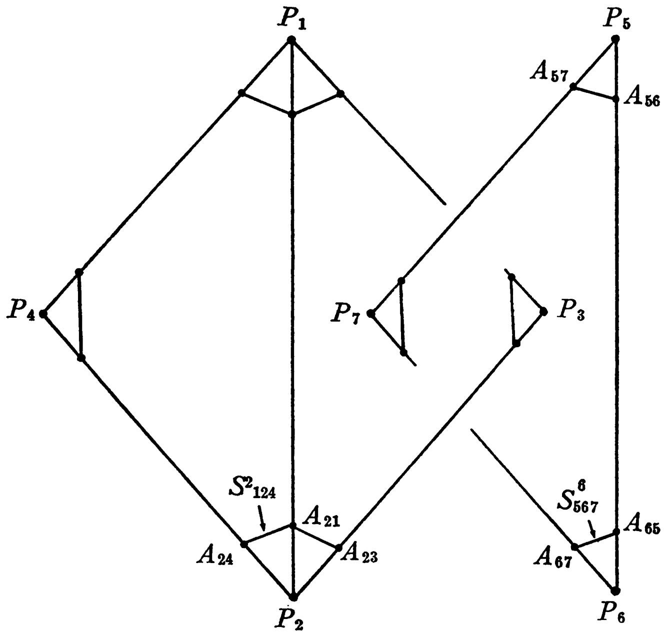

FIGURE 1

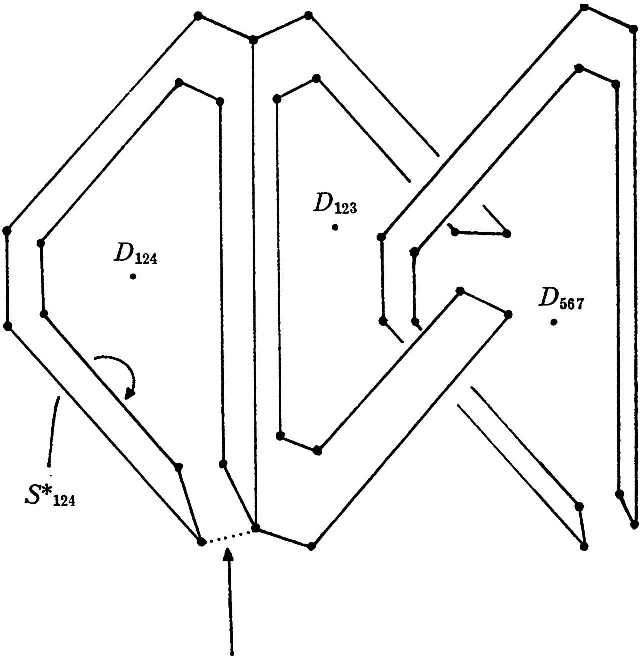

FIGURE 2

but where we deal with $C_3^0 \vee \cdots \vee C_3^0$ ($n+1$ times) instead of C^n_{2n+3}, and make the obvious corrections for the notations. A similar result can be obtained in the same way, involving any known n-complex which is not embeddable in E^{2n}.

REFERENCES

1. R. H. Bing and M. L. Curtis, *Imbedding decompositions of E^3 in E^4*, Proc. Amer. Math. Soc. **11** (1960), 149–155.

2. A. Flores, *Über die Existenz n-dimensionaler Komplexe, die nicht in den R_{2n} topologisch einbettbar sind*, Ergeb. Math. Kolloq. **5** (1932/33), 12–24.

3. D. S. Gillman, *A five circle decomposition of 3-space*, Notices Amer. Math. Soc. **13** (1966), 594 (Abstract 636–65).

4. B. Grünbaum, *Convex polytopes*, Wiley, New York, 1967.

5. L. V. Keldyš, *Some theorems on topological imbedding*, General topology and its relations to modern analysis and algebra, Proc. Sympos. Prague, 1961, Academic Press, New York, 1962; pp. 230–234.

6. R. H. Rosen, *Decomposing 3-spaces into circles and points*, Proc. Amer. Math. Soc. **11** (1960), 918–928.

UNIVERSITY OF WASHINGTON