AN EXTENSION OF BANACH'S MAPPING THEOREM

RICHARD A. BRUALDI[1]

The following mapping theorem of Banach [1] is well known. It is the basis of most proofs of the Schroder-Bernstein equivalence theorem.

If X and Y are sets and $f: X \to Y$ and $g: Y \to X$ are injective mappings, then there exists partitions[2] $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ such that $f(X_1) = Y_1$ and $g(Y_2) = X_2$.

The conclusion of this theorem can be rephrased in the following way.

Let $\Lambda \subseteq X \times Y$ be the relation between X and Y defined by

$$\Lambda = \Lambda_f \cup \Lambda_{g^{-1}}$$

where

$$\Lambda_f = \{(x, f(x)) : x \in X\}, \qquad \text{the graph of } f,$$

and

$$\Lambda_{g^{-1}} = \{(g(y)), y) : y \in Y\}, \qquad \text{the graph of } g^{-1}.$$

Then there exists a bijection $h: X \to Y$ with graph $\Lambda_h \subseteq \Lambda$. With this, we are now prepared to state the main result of this paper which extends the above theorem of Banach. For sets U and V, $U \backslash V$ is the set consisting of those elements of U which are not in V.

1. **Theorem.** *Let X and Y be sets with given partitions $\sum_{i \in I} X_i$ and $Y = \sum_{j \in J} Y_j$ where I and J are arbitrary index sets. For each $i \in I$, let integers a_i and a_i' be specified with $0 \leqq a_i \leqq a_i'$. For each $j \in J$, let integers b_j and b_j' be specified with $0 \leqq b_j \leqq b_j'$. Suppose $f: X^0 \to Y^0$ is a bijection where $X^0 \subseteq X$ and $Y^0 \subseteq Y$ with*

$$a_i \leqq |X_i \backslash X^0| \qquad (i \in I), \tag{1}$$

$$|Y_j \backslash Y^0| \leqq b_j' \qquad (j \in J), \tag{2}$$

and suppose $g: {}^0Y \to {}^0X$ is a bijection where ${}^0Y \subseteq Y$ and ${}^0X \subseteq X$ with

$$|X_i \backslash {}^0X| \leqq a_i' \qquad (i \in I), \tag{3}$$

$$b_j \leqq |Y_j \backslash {}^0Y| \qquad (j \in J). \tag{4}$$

Let $\Lambda = \Lambda_f \cup \Lambda_{g^{-1}}$ where Λ_f and $\Lambda_{g^{-1}}$ are the graphs of f and g^{-1} respec-

Received by the editors November 2, 1967.

[1] Partially supported by National Science Foundation Grant No. GP-7073.

[2] $X = X_1 + X_2$ is a partition of X if $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$.

tively. Then there exist $X' \subseteq X$ *and* $Y' \subseteq Y$ *with*

$$a_i \leqq |X_i \backslash X'| \leqq a_i' \qquad (i \in I), \tag{5}$$

$$b_j \leqq |Y_j \backslash Y'| \leqq b_j' \qquad (j \in J), \tag{6}$$

and a bijection $h: X' \to Y'$ *with graph* $\Lambda_0 \subseteq \Lambda$.

REMARKS. (a) $X = \sum_{i \in I} X_i$ is a *partition* means $X = \bigcup_{i \in I} X_i$ and $X_i \cap X_{i'} = \emptyset$ for all $i \neq i'$ in I.

(b) It is possible that the cardinalities $|X_i \backslash X^0|$ and $|Y_j \backslash {}^0Y|$ be infinite.

(c) If $I = \{1\}$ and $J = \{1\}$ so that the partitions of X and Y are trivial and if $a_1 = a_1' = b_1 = b_1' = 0$, then the above theorem reduces to Banach's mapping theorem. For under these circumstances $g^{-1}: X \to Y$ and $f^{-1}: Y \to X$ are injective mappings and $X' = X$, $Y' = Y$.

2. **Proof of the theorem.** If we restrict the injective mapping f to a subset of X^0, then condition (1) will still be satisfied. Hence we may assume at the start that

$$b_j \leqq |Y_j \backslash Y^0| \leqq b_j' \qquad (j \in J).$$

Let $i \in I$. If $a_i \leqq |X_i \backslash X^0| \leqq a_i'$, then define $X_i^1 = \emptyset$. If on the other hand $|X_i \backslash X^0| > a_i'$, then from (3) we conclude that $|(X_i \backslash X^0) \backslash {}^0X| \leqq a_i'$, so that there exists $X_i^1 \subseteq (X_i \backslash X^0) \cap {}^0X$ such that

$$a_i \leqq |X_i \backslash (X^0 \cup X_i^1)| \leqq a_i'.$$

Define X^1 and P^1 by $X^1 = P^1 = \bigcup_{i \in I} X_i^1$ and define $U^1 = \emptyset$, so that

$$a_i \leqq |X_i \backslash ((X^0 \backslash U^1) \cup P^1)| \leqq a_i' \qquad (i \in I).$$

Note that $P^1 \subseteq {}^0X \backslash X^0$. If $P^1 = \emptyset$, then $X' = X^0$, $Y' = Y^0$, $h = f$ satisfy the conclusion of the theorem. Hence we can assume $P^1 \neq \emptyset$. We proceed inductively. Let $n \geqq 1$ and suppose the following six families of sets have been defined, each family consisting of mutually disjoint sets:

$$\{X^k \subseteq X : 1 \leqq k \leqq n\} \quad \text{with } X^1 \subseteq {}^0X \backslash X^0 \text{ and } X^k \subseteq X^0 \ (1 < k \leqq n), \tag{7}$$

$$\{P^k \subseteq {}^0X \backslash X^0 : 1 \leqq k \leqq n\}, \tag{8}$$

$$\{U^k \subseteq X^0 \backslash {}^0X : 1 \leqq k \leqq n\}, \tag{9}$$

$$\{Y^k \subseteq {}^0Y : 1 \leqq k < n\}, \tag{10}$$

$$\{Q^k \subseteq {}^0Y \backslash Y^0 : 1 \leqq k < n\}, \tag{11}$$

$$\{V^k \subseteq Y^0 \backslash {}^0Y : 1 \leqq k < n\}, \tag{12}$$

with

$$a_i \leqq \left| X_i \backslash \left(\left(X^0 \backslash \bigcup_{k=1}^{n} U^k \right) \cup \bigcup_{k=1}^{n} P^k \right) \right| \leqq a_i' \qquad (i \in I)$$

and

$$b_j \leqq \left| Y_j \backslash \left(\left(Y^0 \backslash \bigcup_{k=1}^{n-1} V^k \right) \cup \bigcup_{k=1}^{n-1} Q^k \right) \right| \leqq b_j' \qquad (j \in J).$$

(If $n=1$, then $\bigcup_{k=1}^{n-1} V^k = \bigcup_{k=1}^{n-1} Q^k = \varnothing$.) We then define

(13) $$Y^n = g^{-1}((X^n \backslash U^n) \cup P^n),$$

(14) $$Q_j^n = (Y^n \cap Y_j) \backslash Y^0 \qquad (j \in J),$$

(15) $$Q^n = \bigcup_{j \in J} Q_j^n \subseteq {}^0Y \backslash Y^0.$$

Since for each $j \in J$, $|Y_j \backslash Y^0| \leqq b_j' < \infty$, Q_j^n is a finite set. If for $j \in J$,

$$b_j \leqq \left| Y_j \backslash \left(\left(Y^0 \backslash \bigcup_{k=1}^{n-1} V^k \right) \cup \bigcup_{k=1}^{n} Q^k \right) \right| \leqq b_j',$$

we define $V_j^n = \varnothing$. Otherwise

$$\left| Y_j \backslash \left(\left(Y^0 \backslash \bigcup_{k=1}^{n-1} V^k \right) \cup \bigcup_{k=1}^{n} Q^k \right) \right| < b_j.$$

But then

$$\left| \left(\left((Y_j \backslash {}^0Y) \backslash \bigcup_{k=1}^{n-1} V^k \right) \cap Y^0 \right) \right|$$
$$\geqq b_j - \left| Y_j \backslash \left(\left(Y^0 \backslash \bigcup_{k=1}^{n-1} V^k \right) \cup \bigcup_{k=1}^{n} Q^k \right) \right|.$$

for *otherwise*

$$|Y_j \backslash {}^0Y| \leqq \left| \left((Y_j \backslash {}^0Y) \backslash \bigcup_{k=1}^{n-1} V^k \right) \cap Y^0 \right|$$
$$+ \left| Y_j \backslash \left(\left(Y^0 \backslash \bigcup_{k=1}^{n-1} V^k \right) \cup \bigcup_{k=1}^{n} Q^k \right) \right| < b_j,$$

which would be a contradiction. Hence we may choose $V_j^n \subseteq ((Y_j \backslash {}^0Y) \backslash \bigcup_{k=1}^{n-1} V^k) \cap Y^0$ with

$$|V_j^n| = b_j - \left| Y_j \backslash \left(\left(Y^0 \backslash \bigcup_{k=1}^{n-1} V^k \right) \cup \bigcup_{k=1}^{n} Q^k \right) \right|.$$

Define $V^n = \bigcup_{j \in J} V_j^n \subseteq Y^0 \backslash {}^0Y$ so that

$$b_j \leqq \left| Y_j \backslash \left(\left(Y^0 \backslash \bigcup_{k=1}^{n} V^k \right) \cup \bigcup_{k=1}^{n} Q^k \right) \right| \leqq b_j' \qquad (j \in J).$$

For $n=1$ each of the families (7), (8), (9) are families of mutually disjoint sets. It then follows from definitions (13), (14) and (15) that if (7), (8) are families of mutually disjoint sets so are $\{Y^k: 1 \leqq k \leqq n\}$ and $\{Q^k: 1 \leqq k \leqq n\}$. Also it follows from the definition of V^n that if $\{V^k: 1 \leqq k < n\}$ is a family of mutually disjoint sets, so is $\{V^k: 1 \leqq k \leqq n\}$.

We now show how to define X^{n+1}, P^{n+1}, and U^{n+1}. Let

$$X^{n+1} = f^{-1}((Y^n \backslash Q^n) \cup V^n), \tag{16}$$

$$U_i^{n+1} = (X^{n+1} \cap X_i) \backslash {}^0X, \tag{17}$$

$$U^{n+1} = \bigcup_{i \in I} U_i^{n+1}. \tag{18}$$

Since for each $i \in I$, $|X_i \backslash {}^0X| \leqq a_i' < \infty$, U_i^{n+1} is a finite set. If for $i \in I$,

$$a_i \leqq \left| X_i \backslash \left(\left(X^0 \backslash \bigcup_{k=1}^{n+1} U^k \right) \cup \bigcup_{k=1}^{n} P^k \right) \right| \leqq a_i',$$

then we define $P_i^{n+1} = \varnothing$. Otherwise

$$\infty > \left| X_i \backslash \left(\left(X^0 \backslash \bigcup_{k=1}^{n+1} U^k \right) \cup \bigcup_{k=1}^{n} P^k \right) \right| > a_i'.$$

But then

$$\left| \left((X_i \backslash X^0) \backslash \bigcup_{k=1}^{n} P^k \right) \cap {}^0X \right|$$
$$\geqq \left| X_i \backslash \left(\left(X^0 \backslash \bigcup_{k=1}^{n+1} U^k \right) \cup \bigcup_{k=1}^{n} P^k \right) \right| - a_i',$$

since *otherwise*

$$\begin{aligned} &| X_i \backslash {}^0X | \\ &\geqq \left\{ | X_i \backslash (X^0 \cup {}^0X) | - \left| \left((X_i \backslash X^0) \backslash \bigcup_{k=1}^{n} P^k \right) \cap {}^0X \right| \right\} \\ &\quad + \left\{ \left| X_i \backslash \left(\left(X^0 \backslash \bigcup_{k=1}^{n+1} U^k \right) \cup \bigcup_{k=1}^{n} P^k \right) \right| - | X_i \backslash {}^0X \cup X^0 | \right\} \\ &> a_i', \end{aligned}$$

which is a contradiction. Hence we can choose $P_i^{n+1} \subseteq ((X_i \backslash X^0) \backslash \bigcup_{k=1}^{n} P^k) \cap {}^0X$ with

$$|P_i^{n+1}| = \left| X_i \backslash \left(\left(X^0 \backslash \bigcup_{k=1}^{n+1} U^k \right) \cup \bigcup_{k=1}^{n} P^k \right) \right| - a_i'.$$

Define $P^{n+1} = \bigcup_{i \in I} P_i^{n+1} \subseteq {}^0X \backslash X^0$ so that

$$a_i \leqq \left| X_i \backslash \left(\left(X^0 \backslash \bigcup_{k=1}^{n+1} U^k \right) \cup \bigcup_{k=1}^{n+1} P^k \right) \right| \leqq a_i' \qquad (i \in I).$$

It follows from the definitions (16), (17) and (18) that if $\{Y^k: 1 \leqq k \leqq n\}$ and $\{Q^k: 1 \leqq k \leqq n\}$ are families of mutually disjoint sets then since $X^1 \subseteq {}^0X \backslash X^0$, $\{X^k: 1 \leqq k \leqq n+1\}$ is a family of mutually disjoint sets. Also since $U^1 = \varnothing$, $\{U^k: 1 \leqq k \leqq n+1\}$ is a family of mutually disjoint sets by (17) and the above. Likewise from the definitions, it follows that $\{P^k: 1 \leqq k \leqq n+1\}$ is a family of mutually disjoint sets. Now define

$$X' = \left(X^0 \backslash \bigcup_{k=1}^{\infty} U^k \right) \cup \bigcup_{k=1}^{\infty} P^k$$

and

$$Y' = \left(Y^0 \backslash \bigcup_{k=1}^{\infty} V^k \right) \cup \bigcup_{k=1}^{\infty} Q^k.$$

Since for each $i \in I$, $|X_i \backslash (X^0 \cup P^1)| < \infty$ and $P_i^k \subseteq X_i \backslash (X^0 \cup P^1)$, $k \geqq 2$, all but a finite number of the P_i^k (i fixed) are empty. Likewise since $U_i^k \subseteq X_i \backslash {}^0X$ and $|X_i \backslash {}^0X| < \infty$, all but a finite number of the U_i^k (i fixed) are empty. Hence

$$a_i \leqq |X_i \backslash X'| \leqq a_i' \qquad (i \in I).$$

In a similar way we derive that

$$b_j \leqq |Y_j \backslash Y'| \leqq b_j' \qquad (j \in J).$$

Let $x \in X'$. If $x \in (X^n \backslash U^n) \cup P^n$ for some $n \geqq 1$, define $h(x) = g^{-1}(x) \in Y^n \subseteq Y'$. If $x \notin (X^n \backslash U^n) \cup P^n$ for any $n \geqq 1$, define $h(x) = f(x) \in Y'$. Suppose for $x_1 \neq x_2$ in X', $h(x_1) = h(x_2)$. Since both f and g are injective, we may assume that $x_1 \in (X^m \backslash U^m) \cup P^m$ for some $m \geqq 1$ and that $x_2 \notin (X^n \backslash U^n) \cup P^n$ for any $n \geqq 1$. Thus $h(x_1) = g^{-1}(x_1) = f(x_2) = h(x_2)$. But then $g^{-1}(x_1) \in Y^m$ and $g^{-1}(x_1) = f(x_2) \notin V^m$. Hence $x_2 \in f^{-1}(Y^m \backslash Q^m) \subseteq X^{m+1}$. If $x_2 \notin {}^0X$, then $x_2 \in U^{m+1}$ and hence $x_2 \notin X'$, a contradiction. If $x_2 \in {}^0X\cdot$ then $x_2 \notin U^{m+1}$ and hence $x_2 \in (X^{m+1} \backslash U^{m+1}) \cup P^{m+1}$, a contradiction. Thus h is injective.

Suppose $y \in Y'$. If $y \in Y^m$ for some $m \geqq 1$ then $g(y) \in (X^m \backslash U^m) \cup P^m \subseteq X'$ and $h(g(y)) = g^{-1}(g(y)) = y$. Otherwise $y \in (Y^0 \backslash \bigcup_{m=1}^{\infty} Y^m) \backslash \bigcup_{k=1}^{\infty} V_k$. Hence $f^{-1}(y) \notin (X^n \backslash U^n) \cup P^n$ for any $n \geqq 1$ and thus $f^{-1}(y) \in X'$

with $h(f^{-1}(y))=f(f^{-1}(y))=y$. Thus $h: X'\to Y'$ is a bijection satisfying the conclusion of the theorem.

3. **Consequences.** The main result of this paper not only contains the original theorem of Banach but also an extension of Banach's theorem given by Knaster and Tarski [**4**] (cf. [**6**, pp. 146–147]) and an essentially equivalent theorem of Perfect and Pym [**5**]. We state it as a corollary, and show how it is a special case.

COROLLARY 1. *Let X and Y be sets. Let $X_1\subseteq X$ and $Y_1\subseteq Y$ with f: $X_1\to Y$ and g: $Y_1\to X$ injective mappings. Let $\Lambda=\Lambda_{f^{-1}}\cup\Lambda_g$. Then there exist sets X', Y' with $X_1\subseteq X'\subseteq X$, $Y_1\subseteq Y'\subseteq Y$ and a bijection h: $Y'\to X'$ with graph $\Lambda_h\subseteq\Lambda$.*

PROOF. Let the set $X\backslash X_1$ be partitioned into its one element subsets, say $X\backslash X_1=\sum_{i\in I}X_i$ where I is an index set with $1\notin I$, so that $X=\sum_{i\in I\cup\{1\}}X_i$ is a partition of X. Likewise, let the set $Y\backslash Y_1$ be partitioned into its one element subsets, say $Y\backslash Y_1=\sum_{j\in J}Y_j$ where J is an index set with $1\notin J$, so that $Y=\sum_{j\in J\cup\{1\}}Y_j$ is a partition of Y. With $i\in I$ associate the integers $b_i=0$, $b_i'=1$, and with 1 associate the integers $b_1=0$, $b_1'=0$. With $j\in J$ associate the integers $a_j=0$, $a_j'=1$, and with 1 associate the integers $a_1=0$, $a_1'=0$. Then f^{-1}: $f(X_1)\to X_1$ is a bijection with

$$|X_1\backslash X_1|\leqq b_1'=0,\qquad |X_i\backslash X_1|\leqq b_i'=1\qquad (i\in I),$$
$$0=a_1\leqq|Y_1\backslash f(X_1)|,\qquad 0=a_j\leqq|Y_j\backslash f(X_1)|\qquad (j\in J).$$

Likewise g^{-1}: $g(Y_1)\to Y_1$ is a bijection with

$$0=b_1\leqq|X_1\backslash g(Y_1)|,\qquad 0=b_i\leqq|X_i\backslash g(Y_1)|\qquad (i\in I),$$
$$|Y_1\backslash Y_1|\leqq a_1'=0,\qquad |Y_j\backslash Y_1|\leqq a_j'=1\qquad (j\in J).$$

Hence by the theorem there exists $X'\subseteq X$, $Y'\subseteq Y$ and a bijection h: $Y'\to X'$ with $\Lambda_h\subseteq\Lambda_{f^{-1}}\cup\Lambda_g$ with, in particular,

$$0=b_1\leqq|X_1\backslash X'|\leqq b_1'=0,\qquad 0=a_1\leqq|Y_1\backslash Y'|\leqq a_1'=0.$$

Hence $X_1\subseteq X'$ and $Y_1\subseteq Y'$. Thus the corollary is a special case of the theorem.

We now give an application of the main result to *transversal theory*. Let $\mathfrak{A}(I)=(A_i: i\in I)$ be a family of subsets of a set E. Here I is an index set, and it is possible that $A_i=A_{i'}$ for $i\neq i'$ in I. A family $(e_i: i\in I)$ is a *system of distinct representatives* of the given family $\mathfrak{A}(I)$ provided $e_i\in A_i$ $(i\in I)$ and the elements e_i $(i\in I)$ are distinct. The set $\{e_i: i\in I\}$ is a *transversal* of $\mathfrak{A}(I)$. If $|E|<\infty$ and $|I|<\infty$ and $E=\sum_{j=1}^{p}E_j$ is a partition of E with associated integers $0\leqq b_j\leqq b_j'$, then A. J. Hoffman and H. W. Kuhn [**3**] gave necessary and sufficient conditions that the family $\mathfrak{A}(I)$ have a transversal $\{e_i: i \in I\}$ with $b_j \leqq |E_j \cap \{e_i: i \in I\}| \leqq b_j'$ $(1 \leqq j \leqq p)$.

A consequence of their conditions, as was noted in [2], is that if there is a transversal $\{e_i' : i \in I\}$ with $b_j \leqq |E_j \cap \{e_i' : i \in I\}|$, $1 \leqq j \leqq p$, and a transversal $\{e_i'' : i \in I\}$ with $|E_j \cap \{e_i'' : i \in I\}| \leqq b_j'$, $1 \leqq j \leqq p$, then there is a transversal with the above properties. In fact the Hoffman-Kuhn theorem is a special case of the so-called symmetric supply-demand theorem [2], which can indeed be used to derive our main result in case the sets X and Y are finite. It is in fact the symmetric supply-demand theorem which led the author to the main result of this paper. The following corollary extends the aspect of the Hoffman-Kuhn theorem mentioned above. It is an immediate consequence of the main result.

COROLLARY 2. *Let* $\mathfrak{A}(I) = (A_i: i \in I)$ *be a family of subsets of a set* E *and let* $E = \sum_{j \in J} E_j$ *be a partition of* E. *Let integers* $0 \leqq b_j \leqq b_j'$ *be given for each* $j \in J$. *Suppose there is a subfamily* $(A_i: i \in I_0)$ *of* $\mathfrak{A}(I)$ *which has a transversal* $\{e_i' : i \in I_0\}$ *with*

$$|E_j \backslash \{e_i' : i \in I_0\}| \leqq b_j' \qquad (j \in J),$$

and suppose $\mathfrak{A}$ *has a transversal* $\{e_i'' : i \in I\}$ *with*

$$b_j \leqq |E_j \backslash \{e_i'' : i \in I\}| \qquad (j \in J),$$

then $\mathfrak{A}$ *has a transversal* $\{e_i: i \in I\}$ *with*

$$b_j \leqq |E_j \backslash \{e_i: i \in I\}| \leqq b_j' \qquad (j \in J).$$

Of course, the complete main result may be translated into the language of transversal theory.

REFERENCES

1. S. Banach, *Un théorème sur les transformations biunivoques*, Fund. Math. **6** (1924), 236–239.

2. L. R. Ford, Jr. and D. R. Fulkerson, *Flows in networks*, Van Nostrand, Princeton, N. J., 1962.

3. A. J. Hoffman and H. W. Kuhn, *On systems of distinct represenatives*, Ann. of Math. Studies, No. 38, Princeton Univ. Press, Princeton, N. J., 1956, pp. 199–206.

4. B. Knaster, *Un théorème sur les fonctions d'ensembles*, Ann. Polon. Math. **6** (1927), 133.

5. Hazel Perfect and J. S. Pym, *An extension of Banach's mapping theorem, with applications to problems concerning common representatives*, Proc. Cambridge Philos. Soc. **62** (1966), 187–192.

6. W. Sierpinski, *Algèbra des ensembles*, Monografie Mat. No. 23, PWN, Warsaw, 1951.

UNIVERSITY OF WISCONSIN