1. Introduction. Let $A$ be a complete discrete valuation ring in characteristic zero with algebraically closed residue field of characteristic $p$. Let $F$ be a one-parameter formal group law defined over $A$. Then there is an injection $c$ of $\text{End}_A F$ onto a subring of $A$. If height $(F)=h<\infty$, $c(\text{End}_A F)$ is a free $\mathbb{Z}_p$-module of rank $\leq h$. We call $F$ almost full over $A$ if $c(\text{End}_A F)$ has rank $h$; in this case all the endomorphisms of $F$ are defined over $A$, and we write simply $\text{End} F$. $F$ is full over $A$ if it is almost full over $A$ and in addition $c(\text{End} F)$ is integrally closed in its fraction field $\mathbb{F}_p$.

The main theorem of the paper by Lubin [2] which contains the above results is a uniqueness theorem: If $F$ and $G$ are both full over $A$ and $c(\text{End} F)=c(\text{End} G)$ (equivalently, $\Sigma_F=\Sigma_G$), then $F$ and $G$ are isomorphic over $A$. I will show that this result is a particular case of a classification theorem for formal groups almost full over $A$.

2. Statement of the theorem. The basic idea is to use the theory of the Tate module $T(F)$, as developed in [3]. (The reader should note that the theorems there remain true under our hypotheses.) $T(F)$ is a free $\mathbb{Z}_p$-module of rank $h$, and is a module over $R=c(\text{End} F)$. $V(F)=T(F)\otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is therefore a vector space over $\Sigma_F$, and must be of dimension one if $F$ is almost full. In this case, $T(F)$ is isomorphic as an $R$-module to a lattice in $\Sigma_F$. Furthermore, by [3, 3.1] $\Sigma$ is the order of this lattice, i.e., $R=\{x\in\Sigma_F: xT(F)\subseteq T(F)\}$.

**Theorem.** Two groups $F$ and $G$ almost full over $A$ with $c(\text{End} F)=c(\text{End} G)=R$ are isomorphic if and only if $T(F)$ and $T(G)$ are isomorphic as $R$-modules. Furthermore, the isomorphism classes of $R$-modules occurring are precisely those of the lattices in $\Sigma_F$ with order $R$.

In particular, the number of nonisomorphic such formal groups is finite and equals the class number of $R$, i.e., the number of isomorphism classes of such lattices.

3. Proof. All formal groups from now on will be almost full over $A$.

**Proposition 3.1 (Lubin).** $F$ and $G$ are isogenous if and only if $\Sigma_F=\Sigma_G$.

**Proof.** The necessity of the condition is [3, 3.0]. Conversely, if $\Sigma_F=\Sigma_G$, $F$ and $G$ are both isogenous to full groups with the same $\Sigma$.
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by [3, 3.2], and these in turn are isomorphic by the uniqueness theorem.

**Lemma 3.2.** There is an $H$ with $c(\text{End } H) = R$ such that $T(H)$ is free of rank one as an $R$-module.

**Proof.** We view $T(F)$ as a lattice in $V(F)$. Let $k$ be the fraction field of $A$, $K$ its algebraic closure. The Galois group $G = \text{Gal } (K/k)$ acts on $T(F)$ and commutes with the action of $A$-endomorphisms, so it acts $\Sigma_F$-linearly on $V(F)$. Since $V(F)$ is one-dimensional over $\Sigma_F$, the action is given by a homomorphism $\rho: G \rightarrow \Sigma_F^\times$. As $T(F)$ is stable under $G$, and $R$ is the order of $T(F)$, we have $\rho(G) \subseteq R$. Hence for any $b \in T(F)$, $Rb$ is a lattice stable under $G$. Therefore it defines an $H$ isogenous to $F$ over $A$, with $T(H)$ isomorphic to $Rb$ as an $R$-module [3, 2.3] and $c(\text{End } H)$ equal to the order of $T(H)$, i.e., $R$.

Now let $L$ be any sublattice of $T(H)$ with order $R$. As before, $\rho(G) \subseteq R$, so $L$ is stable under $G$; hence $L$ defines a formal group $G$ isogenous to $H$ over $A$ and having $T(G) \cong L$. Conversely, if $G$ is any formal group with $c(\text{End } G) = R$, then as we saw $G$ is isogenous to $H$ and hence [3, 2.2] occurs as the group associated to some such lattice $L$. Now since $T(H)$ is free of rank one over $R$, the lattices in it with order $R$ correspond precisely to the ideals $I$ of $R$ with order $R$. We now must prove that such ideals are isomorphic if and only if the corresponding groups are; this will complete the proof of the theorem, since any lattice in $\Sigma_H$ is isomorphic (under multiplication by an integer) to one lying in $R$.

Suppose first that $I$ and $J$ are isomorphic $R$-modules. As the isomorphism extends to a $\Sigma_H$ vector space automorphism of $\Sigma_H$, it is given by a scalar multiplication, so $I = \lambda J$. If $\lambda \in R$, let $f: G \rightarrow H$ be the isogeny with $f(T(G)) = J T(H)$. Then $[\lambda]_H \circ f: G \rightarrow H$ is an isogeny with $[\lambda]_H \circ f(T(G)) = [\lambda]_H(J T(H)) = \lambda J T(H) = I T(H)$, and so $G$ serves also as the formal group corresponding to $I$. If $\lambda \not\in R$, choose an integer $n$ with $n\lambda \in R$; then the group for $I$ is isomorphic to that for $nI = (n\lambda)J$, which is in turn isomorphic to that for $J$.

Conversely, suppose the groups for $I$ and $J$ are isomorphic. Then we have a $G$ and two isogenies $f, g: G \rightarrow H$ with $f(T(G)) = I T(H), \ g(T(G)) = J T(H)$. Replacing $I$ by an integer multiple of itself if necessary, we may assume $I \subseteq J$. Then by [3, 2.2] there is an isogeny $h: G \rightarrow G$ such that $f = g \circ h$. If $h = [x]_G$ for $x \in R$, then $g \circ h = [x]_G \circ g$, since $c$ of these two maps is the same (see [2, 2.1.1]). Hence $I T(H) = f(T(G)) = x g(T(G)) = x J T(H)$. As $T(H)$ is free, $I = x J$ and the modules are isomorphic.

4. Remarks. 1. The uniqueness theorem of Lubin is of course a
particular case of our theorem, since the unique integrally closed order in a local field $\Sigma$ is a discrete valuation ring and hence has just one ideal class. We have not, however, given an independent proof of that theorem; it was used in the proof of Proposition 3.1.

2. Our theorem assumes that $R$ occurs as $c(\text{End } F)$ for some $F$ defined over $A$. Lubin has shown in [3, 3.2] that for any $R$ there is some $A$ over which $R$ occurs, but for a fixed $A$ not all $R$ are possible. To be more specific, let us fix $\Sigma$ and take the unique group $F$ with $c(\text{End } F)$ maximal in $\Sigma$; this group is full over $c(\text{End } F)$ [4, Theorem 1], and so is available whenever $k$ contains $\Sigma$. The argument of Lemma 3.2 applied to $F$ shows then that those $R$ which occur are precisely those which contain the group $\rho(\mathfrak{g})$.

3. If we take the integral closure of $A$ in a finite extension of $k$, we get another ring satisfying our hypotheses. This extension reduces $\rho(\mathfrak{g})$ and thus makes more orders possible, but the theorem implies that it can have no other effect. Once a single group $F$ with $c(\text{End } F) = R$ is defined over $A$, so are all the others; expanding $k$ cannot introduce any new isomorphism types, nor can any nonisomorphic groups become isomorphic over the larger ring.

4. The technique used here resembles that of a standard theorem on the isomorphism types of elliptic curves with complex multiplication. Serre [5] has drawn attention to the fact that that result can be interpreted as saying that the isomorphism classes of curves with endomorphism ring $R$ are a principal homogeneous space over the group of rank one projective $R$-modules. Similar results have been proved for certain abelian varieties when the endomorphism ring is a maximal order. It may therefore be useful to point out that no such formulation is possible here: the modules occurring in the theorem need not be projective, nor need they form a group. (An example is easily constructed following Exercise 18 [1, p. 94].) A similar example shows that the use of Lemma 3.2 cannot be avoided; there is an almost full group $G$ and ideals $I \neq J$ of $c(\text{End } G)$ with $IT(G) = JT(G)$.

**References**


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