# ON THE SEMISIMPLICITY OF MODULAR GROUP ALGEBRAS

D. S. PASSMAN

Let $G$ be a discrete group, let $K$ be a field and let $KG$ denote the group algebra of $G$ over $K$. We say that $KG$ is semisimple if its Jacobson radical $JKG$ is zero. If $K$ has characteristic 0 and $K$ is not an algebraic extension of the rationals then it is known [**1**, Theorem 1] that $KG$ is semisimple. Moreover it appears likely that in the remaining characteristic 0 cases we also have semisimplicity. Thus nothing particularly interesting occurs here.

If $K$ has characteristic $p>0$ and $G$ is a $p'$-group (that is, $G$ has no elements of order $p$), then it is known (see [**2**]) that $KG$ has no nil ideals and that for suitably big fields the group algebra $KG$ is semisimple. Again it appears that for the remaining fields we also have semisimplicity.

The interest in characteristic $p$ stems from the fact that, unlike the case in which $G$ is finite, it is quite possible for $G$ to have elements of order $p$ and yet have the group algebra $KG$ be semisimple. Several examples of such groups have been exhibited and in each case a big abelian group is involved as either a subgroup or a factor group.

In this paper we study the group algebras $KG$ of those groups $G$ having a big abelian subgroup or factor group. The methods used are extremely elementary. Two interesting examples of the type of groups which we can deal with are as follows. Let $P$ be a cyclic group of order $p$ and let $C$ be an infinite cyclic group. Let $G_1=C \wr P$ and $G_2=P \wr C$ where $\wr$ denotes the restricted Wreath product. Now $G_1$ has a normal torsion free abelian subgroup of finite index $p$ and $G_2$ has a normal elementary abelian $p$-subgroup $E$ with $G_2/E$ infinite cyclic. Surprising as it may seem if $K$ is an algebraically closed field of characteristic $p$, then $KG_1$ and $KG_2$ are both semisimple.

For the remainder of this paper $K$ is an algebraically closed field of characteristic $p$. By a linear $K$-character of $KG$ we mean a $K$-homomorphism $\lambda: KG\rightarrow K$.

LEMMA 1. (AMITSUR). *If $H$ is a subgroup of $G$ then*

$$(JKG) \cap (KH) \subseteq JKH.$$

PROOF. [**2**, Lemma 10].

Received by the editors February 12, 1968.

LEMMA 2. *Let $A$ be an abelian $p'$-group.*

(i) *Let $a_1, a_2, \cdots, a_n$ be $n$ distinct elements of $A$. Then there exists $n$ linear $K$-characters $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $KA$ such that the matrix $[\lambda_i(a_j)]$ is nonsingular.*

(ii) $\bigcap_\lambda \ker \lambda = \{0\}$ *where $\lambda$ runs over all linear $K$-characters. In particular $KA$ is semisimple.*

PROOF. (i) We make a series of observations.

(1) Suppose $\{a_1, \cdots, a_n\} \subseteq \{b_1, \cdots, b_m\}$ and there exist linear $K$-characters $\lambda_1, \cdots, \lambda_m$ with $[\lambda_i(b_j)]$ nonsingular. Then there exists a suitable subset $\{\lambda_{r_1}, \lambda_{r_2}, \cdots, \lambda_{r_n}\}$ with $[\lambda_{r_i}(a_j)]$ nonsingular. This follows since the submatrix of $[\lambda_i(b_j)]$ composed of the columns corresponding to the set $\{a_j\}$ has rank $n$ and hence a suitable $n \times n$ submatrix of it is nonsingular.

(2) If (i) holds for $A$ and $B$, then it holds for $A \times B$. In view of (1) we can assume that the given subset of $A \times B$ is a product set, that is $\{a_ib_j \mid a_i \in A,\ b_j \in B\}$. Choose linear $K$-characters $\lambda_1, \cdots, \lambda_n$ of $KA$ with $[\lambda_r(a_i)]$ nonsingular and choose linear $K$-characters $\mu_1, \cdots, \mu_m$ of $KB$ with $[\mu_s(b_j)]$ nonsingular. Then $\{\lambda_r\mu_s\}$ is a set of linear $K$-characters of $K(A \times B)$ and the matrix $[\lambda_r\mu_s(a_ib_j)] = [\lambda_r(a_i)] \times [\mu_s(b_j)]$ is the Kronecker product of two nonsingular matrices and hence is nonsingular.

(3) We show that (i) holds if $A = \langle x \rangle$ is cyclic. Suppose first that $A$ is infinite. By (1) we can assume that $\{a_1, \cdots, a_n\} = \{x^i \mid |i| \leq r\}$. Since $K$ is infinite we can choose $2r+1$ distinct nonzero elements $w_i \in K$ for $i = -r, -r+1, \cdots, 0, \cdots, r$. Define $\lambda_i \colon KA \to K$ by $\lambda_i(x) = w_i$. Then $[\lambda_i(x^j)] = [w_i^j]$ is a modified Vandermonde matrix with distinct entries and hence is nonsingular. Now let $A$ be finite of order $r$. By (1) we can assume that $\{a_1, \cdots, a_n\} = A$. Since $(r, p) = 1$ by assumption $K$ contains $r$ distinct $r$th roots of unity $w_1, \cdots, w_r$. Again define $\lambda_i \colon KA \to K$ by $\lambda_i(x) = w_i$. Then $[\lambda_i(x^j)] = [w_i^j]$ is also nonsingular.

(4) We prove (i). Let $\{a_1, \cdots, a_n\} \subseteq A$ and let $B$ be the subgroup of $A$ generated by this subset. Then $B$ is a finitely generated abelian group and hence a direct product of cyclic groups. By (2), (3) and induction we can find linear $K$-characters $\lambda_i \colon KB \to K$ such that $[\lambda_i(a_j)]$ is nonsingular. Since $K$ is algebraically closed a Zorn's lemma argument shows that the $\lambda_i$ can be extended to linear $K$-characters of $KA$. Hence (i) follows.

(ii) Let $\alpha = \sum_1^n k_ia_i \in \bigcap_\lambda \ker \lambda$. Choose $\lambda_1, \cdots, \lambda_n$ as in (i) with $[\lambda_i(a_j)]$ nonsingular. Then for $k = 1, 2, \cdots, n$, $0 = \lambda_i(\alpha) = \sum_{j=1}^n k_j\lambda_i(a_j)$ and hence $k_1 = \cdots = k_n = 0$ and $\alpha = 0$. This completes the proof of the lemma.

THEOREM 3. *Let $G$ have a normal abelian subgroup $A$ of index $n$ and let $K$ be an algebraically closed field of characteristic $p$.*

(i) *If $A$ is a $p'$-group, then $(JKG)^n = \{0\}$.*

(ii) *If the Sylow $p$-subgroup of $A$ is finite, then $JKG$ is nilpotent.*

(iii) *$JKG$ is locally nilpotent.*

(iv) *$JKG \neq \{0\}$ if and only if $G$ has an element $g$ of order $p$ with $[A:\mathfrak{C}_A(g)] < \infty$.*

PROOF. (i) Let $\lambda: KA \to K$ be a linear $K$-character of $KA$ and let $\lambda^G$ be the induced representation. Thus $\lambda^G: KG \to K_n$ where $K_n$ is the ring of $n \times n$ matrices over $K$. Clearly $\ker \lambda^G \subseteq (\ker \lambda)(KG)$ and since $KG$ is free over $KA$, $\bigcap_\lambda \ker \lambda^G \subseteq (\bigcap_\lambda \ker \lambda)(KG)$. Thus by Lemma 2 (ii), $\bigcap_\lambda \ker \lambda^G = \{0\}$. Now the image of $JKG$ under $\lambda^G$ is a radical $K$-subalgebra of $K_n$ and hence $(JKG)^n \subseteq \ker \lambda^G$. Thus $(JKG)^n \subseteq \bigcap_\lambda \ker \lambda^G = \{0\}$ and (i) follows.

(ii) Let $P$ be the Sylow $p$-subgroup of $A$ so that $P$ is finite and normal in $G$. Let $N = JKP$ so that $N$ is nilpotent and $KP/N \simeq K$. If $g \in G$ then $gN = Ng$ and hence the ideal $N(KG)$ is nilpotent and $JKG \supseteq N(KG)$. Now $KG/N(KG) \simeq K(G/P)$ and $A/P$ is a normal abelian $p'$-subgroup of $G/P$. Hence by (i) $JKG/N(KG) = JK(G/P)$ is nilpotent and (ii) follows.

(iii) Let $\alpha_1, \cdots, \alpha_m \in JKG$. We can find a finitely generated subgroup $H$ of $G$ with $\alpha_1, \cdots, \alpha_n \in KH$. By Lemma 1, $\alpha_1, \cdots, \alpha_m \subseteq JKH$. Now $B = A \cap H$ is a normal abelian subgroup of $H$ of index $\leqq n$ and since $H$ is finitely generated so is $B$. Hence the Sylow $p$-subgroup of $B$ is finite and by (ii) $JKH$ is nilpotent. This clearly yields (iii).

(iv) Suppose first that $G$ has an element $g$ of order $p$ with $[A:\mathfrak{C}_A(g)] < \infty$. Then by Dietzmann's lemma [2, Lemma 9] $G$ has a finite normal subgroup whose order is divisible by $p$. Hence [2, Theorem III] $G$ has a nonzero nilpotent ideal so $JKG \neq \{0\}$. Conversely let $JKG \neq \{0\}$. If $A$ has an element $g$ of order $p$, then $[A:\mathfrak{C}_A(g)] = 1$, and the result follows. If $A$ is a $p'$-group, then by (i) $JKG$ is nilpotent. Hence [2, Theorem III] $G$ has a finite normal subgroup $H$ whose order is divisible by $p$. If $g \in H$ has order $p$ then all conjugates of $g$ are contained in $H$ so $[G:\mathfrak{C}_G(g)] < \infty$ and hence $[A:\mathfrak{C}_A(g)] < \infty$. This completes the proof of the theorem.

COROLLARY 4. *Let $B$ be an infinite abelian $p'$-group and let $H$ be finite. Set $G = B \wr H$. If $K$ is an algebraically closed field of characteristic $p$, then $KG$ is semisimple.*

PROOF. Suppose $|H| = n$. Then $G$ has a normal abelian subgroup $A$ of index $n$ with $A = B_1 + B_2 + \cdots + B_n$, the direct sum of $n$ copies of

$B$. Thus $A$ is a $p'$-group. If $g$ is an element of $G$ of order $p$ then $g \notin A$ so $g$ acts on $A$ by permuting the summands in cycles of length $p$. Since $B$ is infinite, this yields $[A : \mathfrak{C}_A(g)] = \infty$ and hence by Theorem 3 (iv) we have $JKG = \{0\}$.

A special case of the above is the result on the group $G_1$ stated in the introduction.

THEOREM 5. *Let $H \triangleleft G$ with $G/H = A$ an abelian $p'$-group. Let $K$ be an algebraically closed field of characteristic $p$. If $I$ is a characteristic ideal of $KG$, then $I = (I \cap KH)(KG)$.*

PROOF. Let $\bar{g}$ denote the image of $g \in G$ in $G/H = A$. Let $\lambda$ be a linear $K$-character of $KA$. We define a $K$-linear map $\Lambda: KG \to KG$ by $\Lambda(g) = \lambda(\bar{g})g$. It is easy to see that $\Lambda$ is in fact an algebra automorphism of $KG$.

Now let $\alpha \in I$. Then we can choose coset representatives $a_1, \cdots, a_n$ of $H$ in $G$ with $\alpha = f_1a_1 + \cdots + f_na_n$ and $f_i \in KH$. Let $\lambda_1, \cdots, \lambda_n$ be linear $K$-characters of $KA$ guaranteed by Lemma 2 (i) with $[\lambda_i(\bar{a}_j)]$ nonsingular. Let $\Lambda_i$ be the automorphism induced by $\lambda_i$ as above. Since $I$ is a characteristic ideal of $KG$ we have

$$\lambda_i(\bar{a}_1)f_1a_1 + \lambda_i(\bar{a}_2)f_2a_2 + \cdots + \lambda_i(\bar{a}_n)f_na_n = \Lambda_i(\alpha) = \alpha_i \in I$$

for $i = 1, 2, \cdots, n$. Since the matrix $[\lambda_i(\bar{a}_j)]$ is nonsingular, we can find field elements $k_{ij}$ with

$$f_ia_i = k_{i1}\alpha_1 + k_{i2}a_2 + \cdots + k_{in}\alpha_n \in I.$$

Thus $f_i = (f_ia_i)a_i^{-1} \in I \cap KH$ and hence $I \subseteq (I \cap KH)(KG)$. Since the reverse inclusion is obvious, the result follows.

THEOREM 6. *Let $H \triangleleft G$ with $G/H = A$ an abelian $p'$-group. Let $K$ be an algebraically closed field of characteristic $p$.* Set $J_0 = (JKG) \cap (KH)$. *Then*

(i) $J_0 \subseteq JKH$,

(ii) $JKG = (J_0)(KG)$.

(iii) *Let $x \in G$ be such that $xH$ is an element of infinite order in $G/H = A$. If $\alpha \in J_0$ then for some integer $n \geq 0$*

$$\alpha\alpha^x\alpha^{x^2} \cdots \alpha^{x^n} = 0$$

*where $\alpha^y = y\alpha y^{-1}$.*

PROOF. (i) and (ii) follow from Lemma 1 and Theorem 5 respectively. We consider (iii). Let $C = \langle x \rangle$ and let $\bar{G} = \langle H, x \rangle = HC$. Then $\alpha x \in (JKG) \cap K\bar{G}$ and hence by Lemma 1, $\alpha x \in JK\bar{G}$. Let $w = \sum \beta_i x^i$ be a quasi-inverse of $\alpha x$ with $\beta_i \in KH$. Then $(\alpha x) + w + (\alpha x)w = 0$ yields

$$\alpha x + \sum \beta_i x^i + \sum \alpha \beta_i^x x^{i+1} = 0.$$

Hence we have

$$(*) \qquad \beta_i = -\alpha \beta_{i-1}^x \qquad \text{for } i \neq 1,$$

$$(**) \qquad \beta_1 = -\alpha - \alpha \beta_0^x.$$

Since $\beta_{-q}=0$ for some $q>0$, equation (*) and induction imply that $\beta_i=0$ for $i \leqq 0$. Hence (**) yields $\beta_1=-\alpha$ and then (*) and induction yield

$$\beta_i = (-)^i \alpha \alpha^x \cdots \alpha^{x^{i-1}}$$

Since $\beta_{n+1}=0$ for some $n \geqq 0$ the result follows.

We remark that the above is a generalization of Theorem 5.2 of [3]. Additional results of this type can be derived by considering expressions other than $\alpha x$. For example we could choose $x_1, \cdots, x_r \in G-H$ such that their images in $A$ are independent elements of infinite order and then consider the quasi-inverse of $\alpha(x_1+\cdots+x_r)$. However (iii) above appears to be the most useful expression.

COROLLARY 7. *Let $W$ be a group, let $A$ be an abelian $p'$-group containing an element of infinite order and let $G=W \wr A$. If $K$ is an algebraically closed field of characteristic $p$, then $KG$ is semisimple.*

PROOF. $G$ has a normal subgroup $H=\Sigma W_a$ which is a direct sum of copies of $W$ indexed by the elements of $A$. Moreover $G/H=A$. By Theorem 6 (ii) it suffices to show that $J_0=(JKG)\cap(KH)$ is trivial. Let $\alpha \in J_0$. Then there exists $a_1, a_2, \cdots, a_m \in A$ with $\alpha \in KB$ where $B=W_{a_1}+W_{a_2}+\cdots+W_{a_m}$. Let $y \in A$ have infinite order and think of $y$ as element of $G$. Choose integer $r$ sufficiently large so that the elements $a_1, \cdots, a_m$ are in distinct cosets of $\langle y^r \rangle$. Then $x=y^r$ has infinite order and by Theorem 6 (iii) there exists an integer $n \geqq 0$ with $\alpha\alpha^x \cdots \alpha^{x^n}=0$. Now $\alpha^{x^i} \in KB^{x^i}$ and by our choice of $x$, $B+B^x+\cdots+B^{x^n}$ is a direct sum. Hence $\alpha=0$ and the result follows.

This yields our introductory remarks about the group $G_2$.

## REFERENCES

1. S. A. Amitsur, *On the semisimplicity of group algebras*, Michigan Math. J. **6** (1959), 251–253.

2. D. S. Passman, *Nil ideals in group rings*, Michigan Math. J. **9** (1962), 375–384.

3. D. A. R. Wallace, *The Jacobson radicals of the group algebras of a group and of certain normal subgroups*, Math. Z. **100** (1967), 282–294.

YALE UNIVERSITY