NOTES ON SUBDIRECT PRODUCTS OF SEMIGROUPS AND RECTANGULAR BANDS

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1. **Introduction.** Let $\{S_\alpha: \alpha \in A\}$ be a family of semigroups. Let p_α be the natural projection from the direct product $\prod\{S_\alpha: \alpha \in A\}$ onto S_α. A subsemigroup D of $\prod\{S_\alpha: \alpha \in A\}$ is called a subdirect product of $\{S_\alpha: \alpha \in A\}$ if $p_\alpha(D) = S_\alpha$ for all $\alpha \in A$.

A congruence ρ on a semigroup S is called a $\mathfrak{P}$-congruence if S/ρ satisfies a property $\mathfrak{P}$. If $\mathfrak{P}$ is, for example, "semilattice," "reductivity" or "rectangular band," there exists the smallest $\mathfrak{P}$-congruence ρ_0 on S. S/ρ_0 is called the greatest $\mathfrak{P}$-homomorphic image of S. A semigroup S is called $\mathfrak{P}$-indecomposable if any $\mathfrak{P}$-homomorphic image Y of S is trivial, $|Y| = 1$, equivalently $\rho_0 = S \times S$.

In this paper we will discuss how to determine all subdirect products D of a semigroup S and a rectangular band, and will find a necessary condition for isomorphism of subdirect products in the case where S is commutative and reductive. We also will prove that the greatest semilattice-homomorphic image of D is isomorphic to that of S.

All concepts not defined can be found in Clifford and Preston's book [1].

2. **Construction of subdirect products.** A rectangular band B is isomorphic to the direct product $L \times R$ of a left zero semigroup L and a right zero semigroup R:

$$L \times R = \{(\lambda, \rho): \lambda \in L, \rho \in R\}, \qquad (\lambda, \rho)(\lambda', \rho') = (\lambda, \rho').$$

A subsemigroup I of a semigroup S is called a bi-ideal of S if $ISI \subseteq I$. (See [1].) Let X be a subset of S. Then $X \cup X^2 \cup XSX$ is the smallest bi-ideal of S containing X. If U is a right ideal and V a left ideal of S, any subset I of S satisfying $UV \subseteq I \subseteq U \cap V$ is a bi-ideal of S. Every bi-ideal of S is obtained in this manner. (See [1, p. 85].)

THEOREM 1. *Let S be a semigroup and $B = L \times R$ be a rectangular band. Let $\mathfrak{J}$ denote the set of all bi-ideals of S. If a mapping $\eta: B \to \mathfrak{J}$ satisfies*

(1) $S = \bigcup\{\eta(a): a \in B\}$ *and*

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(2) $\eta(a)\cdot\eta(b)\subseteq\eta(ab)$

for all $a, b\in B$, *then the set* $D=\{(x, a): a\in B$ *and* $x\in\eta(a)\}$ *is a subdirect product of* S *and* B. *Furthermore, every subdirect product of* S *and* B *is obtained in this manner.*

PROOF. Let $\eta: B\to\mathfrak{J}$ satisfy (1) and (2) and define $D\subseteq S\times B$ as above. If $(s, a), (t, b)\in D$, then $(st, ab)\in D$ by (2). Hence D is a subsemigroup. Since $\eta(a)\neq\varnothing$ for all $a\in B$ and since (1) holds, D is a subdirect product.

Conversely, suppose $D\subseteq S\times B$ is a subdirect product. Define a map η from B into the power set of S by $\eta(a)=\{x: (x, a)\in D\}$. Since $bcb=b$ for all $b, c\in B$, η obviously takes values in $\mathfrak{J}$. The map η satisfies (1) since each $x\in S$ appears in some $\eta(a)$. It also satisfies (2), for if $x\in\eta(a)$ and $y\in\eta(b)$, $(x, a)(y, b)=(xy, ab)$ or $xy\in\eta(ab)$. Finally, D can be recovered from η by noticing that $D=\{(x, a): a\in B$ and $x\in\eta(a)\}$. The proof is complete.

For the following theorem we write $(x; \lambda, \rho)$ for the element $(x, a)\in S\times B$ where $a=(\lambda, \rho)$ in $B=L\times R$.

THEOREM 2. *Let* S *be a regular semigroup and* $B=L\times R$ *be a rectangular band. Let* $\mathfrak{L}$ *denote the set of all left ideals of* S *and* $\mathfrak{R}$ *denote the set of all right ideals of* S. *Let* $\phi: L\to\mathfrak{R}$ *and* $\psi: R\to\mathfrak{L}$ *be maps such that*

(3) $\bigcup\{\phi(\lambda): \lambda\in L\}=\bigcup\{\psi(\rho): \rho\in R\}=S.$

Then the map $\eta: B\to\mathfrak{J}$ *defined by* $\eta(\lambda, \rho)=\phi(\lambda)\cap\psi(\rho)$ *satisfies* (1) *and* (2) *of Theorem* 1. *Conversely, every map* η *of Theorem* 1 *is obtained in this manner.*

PROOF. Let ϕ and ψ satisfy (3) and define η by $\eta(\lambda, \rho)=\phi(\lambda)\cap\psi(\rho)$, $(\lambda, \rho)\in B$. The set $\eta(\lambda, \rho)$ is obviously a bi-ideal of S. Also,

$$\begin{aligned}\bigcup\{\eta(\lambda, \rho): (\lambda, \rho)\in B\} &= \bigcup\{\phi(\lambda)\cap\psi(\rho): (\lambda, \rho)\in B\}\\ &= (\bigcup\{\phi(\lambda): \lambda\in L\})\cap(\bigcup\{\psi(\rho): \rho\in R\})\\ &= S\cap S = S,\end{aligned}$$

and

$$\eta(\lambda, \rho)\cdot\eta(\lambda', \rho')\subseteq\phi(\lambda)\cdot\psi(\rho')\subseteq\phi(\lambda)\cap\psi(\rho')=\eta(\lambda, \rho')$$

for all $(\lambda, \rho), (\lambda', \rho')\in B$. Thus η satisfies (1) and (2) of Theorem 1.

Conversely, let η satisfy (1) and (2) of Theorem 1. For $\lambda\in L$ and $\rho\in R$ define $\phi(\lambda)=\bigcup\{\eta(\lambda, \rho): \rho\in R\}$ and $\psi(\rho)=\bigcup\{\eta(\lambda, \rho): \lambda\in L\}$. By (1) and (2),

$$\begin{aligned}\phi(\lambda)\cdot S &= (\cup\{\eta(\lambda,\rho):\rho\in R\})\cdot(\cup\{\eta(\lambda',\rho'):(\lambda',\rho')\in B\})\\ &= \cup\{\eta(\lambda,\rho)\cdot\eta(\lambda',\rho'):\rho\in R,(\lambda',\rho')\in B\}\\ &\subseteq \cup\{\eta(\lambda,\rho'):\rho'\in R\} = \phi(\lambda).\end{aligned}$$

Hence $\phi: L\to\mathfrak{R}$. A similar argument shows that $\psi: R\to\mathfrak{L}$. By (1), ϕ and ψ obviously satisfy (3). We need to show that $\eta(\lambda,\rho)=\phi(\lambda)\cap\psi(\rho)$. Clearly $\eta(\lambda,\rho)\subseteq\phi(\lambda)\cap\psi(\rho)$. So suppose $x\in\phi(\lambda)\cap\psi(\rho)$. Then $x\in\eta(\lambda,\rho')$ and $x\in\eta(\lambda',\rho)$ for some $(\lambda',\rho')\in B$. But there exists $a\in S$ such that $xax=x$. Hence, if $a\in\eta(\lambda'',\rho'')$,

$$x = xax \in \eta(\lambda,\rho')\eta(\lambda'',\rho'')\eta(\lambda',\rho)\subseteq\eta(\lambda,\rho)$$

by (2). This completes the proof.

3. **Further results.** Let S_i be a semigroup, B_i a rectangular band, and D_i a subdirect product of S_i and B_i $(i=1,2)$. If $D_1\cong D_2$, it is not necessarily true that $S_1\cong S_2$ and $B_1\cong B_2$. We can, however, conclude this if the semigroups S_1 and S_2 are commutative and reductive.

A semigroup S is called reductive if $axb=ayb$ for all $a,b\in S$ implies $x=y$. If S is commutative, this condition is equivalent to: $ax=ay$ for all $a\in S$ implies $x=y$.

THEOREM 3. *Let D_i be a subdirect product of a rectangular band B_i and a semigroup S_i which is commutative and reductive* $(i=1,2)$. *If $D_1\cong D_2$, then $B_1\cong B_2$ and $S_1\cong S_2$.*

PROOF. In this proof, p_{S_i} and p_{B_i} denote the natural projections of $S_i\times B_i$ onto S_i and B_i respectively $(i=1,2)$. First we will prove that for $x,y\in D_i$

(4) $p_{B_i}(x)=p_{B_i}(y)$ if and only if $xy=yx$,

(5) $p_{S_i}(x)=p_{S_i}(y)$ if and only if $axb=ayb$ for all $a,b\in D_i$.

If $xy=yx$, then $p_{B_i}(x)p_{B_i}(y)=p_{B_i}(xy)=p_{B_i}(yx)=p_{B_i}(y)p_{B_i}(x)$ which implies $p_{B_i}(x)=p_{B_i}(y)$ since B_i is a rectangular band. The converse is immediate because S is commutative. This proves (4). To prove (5), suppose $axb=ayb$ for all $a,b\in D_i$. Then we have

$$p_{S_i}(a)p_{S_i}(x)p_{S_i}(b) = p_{S_i}(a)p_{S_i}(y)p_{S_i}(b) \qquad \text{for all } a,b\in D_i.$$

This implies $p_{S_i}(x)=p_{S_i}(y)$ since p_{S_i} is onto and S is reductive. The converse is obvious. Thus we have (5).

If we define $x\beta_i y$ by $p_{B_i}(x)=p_{B_i}(y)$ and $x\gamma_i y$ by $p_{S_i}(x)=p_{S_i}(y)$, then (4) and (5) show that β_i and γ_i are not in terms of the projections p_{B_i} and p_{S_i} respectively. It is easy to see that β_i is the smallest rectangular-band congruence on D_i and γ_i is the smallest reductive congruence on D_i, and $D_i/\beta_i \cong B_i$, $D_i/\gamma_i \cong S_i$. Since $D_1 \cong D_2$, we have $B_1 \cong B_2$ and $S_1 \cong S_2$.

REMARK. In (4) we have used only commutativity of S. Accordingly if S is commutative and if $D_1 \cong D_2$, then $B_1 \cong B_2$.

THEOREM 4. *Let S be a semigroup and B a rectangular band. The greatest semilattice-homomorphic image of a subdirect product of S and B is isomorphic to that of S.*

PROOF. Let D be a subdirect product of S and B. Every element of D is denoted by (t, a), $t \in S$, $a \in B$. Let σ be the congruence on D defined by

$$(t, a)\sigma(s, b) \qquad \text{if and only if } t = s.$$

If ρ is any semilattice-congruence ($\mathcal{S}$-congruence) on D, we will show that $\sigma \subseteq \rho$. To do this we will use the fact that $xyx = x$ for elements of a rectangular band. Let (t, a), $(t, b) \in D$.

$$\begin{aligned} (t, a)\rho(t^4, ab^2a) &= (t, a)(t, b)(t, b)(t, a) \\ \rho(t, b)(t, a)(t, a)(t, b) &= (t^4, ba^2b)\rho(t, b). \end{aligned}$$

Hence $\sigma \subseteq \rho$ for all $\mathcal{S}$-congruences ρ. Each ρ induces an $\mathcal{S}$-congruence $\bar{\rho}$ on D/σ; conversely, an $\mathcal{S}$-congruence $\bar{\rho}$ on $S \cong D/\sigma$ induces an $\mathcal{S}$-congruence ρ on D with $\sigma \subseteq \rho$. Therefore the smallest $\mathcal{S}$-congruence ρ_0 on D corresponds to the smallest $\mathcal{S}$-congruence $\bar{\rho}_0$ on S in the above sense. Consequently the greatest $\mathcal{S}$-homomorphic image of D is isomorphic to that of S.

COROLLARY. *If S is $\mathcal{S}$-indecomposable then any subdirect product of S and B is $\mathcal{S}$-indecomposable. Conversely if one subdirect product of S and B is $\mathcal{S}$-indecomposable, then S is also.*

REFERENCE

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

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