A NOTE ON THE LOCAL COEFFICIENT PROBLEM

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The Bieberbach conjecture asserts that if $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is schlicht in the unit disk, then $\operatorname{Re} a_n \leqq n$, with equality holding only for the Koebe function

$$K(z) = z + \sum_{n=2}^{\infty} n z^n$$

or one of its rotations. The conjecture was proved for $n=2$, 3, and 4 by Bieberbach [1], Loewner [2] and Garabedian and Schiffer [3], respectively. In recent papers, it has been proved to be true if f is sufficiently close to K in one or more of the topologies defined by

$$\sum_{k=2}^{n} | a_k - k |, \tag{1}$$

$$\operatorname{Re}(2 - a_2), \tag{2}$$

$$\operatorname{Re}(3 - a_3). \tag{3}$$

Garabedian, Ross and Schiffer [4] proved the local result for even n in the topology (1). Garabedian and Schiffer [5] complemented this result by proving that for odd n it is true in the topology defined by (2). They also indicated how their method should be modified to obtain a similar conclusion for even n. Bombieri [6] gave independent proofs by showing that

$$\liminf_{a_2 \to 2} \frac{n - \operatorname{Re}(a_n)}{2 - \operatorname{Re}(a_2)} > 0 \qquad \text{if } n \text{ is even,}$$

and

$$\liminf_{a_3 \to 3} \frac{n - \operatorname{Re}(a_n)}{3 - \operatorname{Re}(a_3)} > 0 \qquad \text{if } n \text{ is odd.}$$

He also showed that

$$\liminf_{a_2 \to 2} \frac{3 - \operatorname{Re}(a_3)}{[2 - \operatorname{Re}(a_2)]^{3/2}} = \frac{8}{3}, \tag{4}$$

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which implies that the topology defined by (2) is no stronger than the one defined by (3).

It is the purpose of this note to point out that the equivalence of the topologies considered above is a simple consequence of Loewner's formulas. The equivalence of (1) and (2) is proved by taking $P_k(\xi) = \xi_k$ in the theorem below and applying (6). Similarly, by choosing $P_3(\xi) = \xi_3$ and using (5) together with Bombieri's inequality (4), the equivalence of (2) and (3) is established. The equivalence of (1) and (3) then follows from transitivity.

THEOREM. *Let $P(\xi_2, \cdots, \xi_n)$ be a polynomial with real coefficients. If $f(z)$ is normalized and schlicht in the unit disk, then there exist constants A and B, independent of f, such that*

$$|\operatorname{Re}(P(a_2, \cdots, a_n) - P(2, \cdots, n))| \leqq A(2 - \operatorname{Re} a_2) \tag{5}$$

and

$$|P(a_2, \cdots, a_n) - P(2, \cdots, n)| \leqq B(2 - \operatorname{Re} a_2)^{1/2}. \tag{6}$$

PROOF. It is a consequence of Loewner's formulas that, for a set of f's which are dense in the nth coefficient region, P can be expressed as a finite sum of the form

$$P(a_2, \cdots, a_n) = \sum b_k \int f_k(t_1, \cdots, t_{m_k}) \exp\left(i \sum_{j=1}^{m_k} \mu_j \phi(t_j)\right) dt_1 \cdots dt_{m_k}.$$

Here, b_k is a real constant, f_k is a real continuous function with support in the m_k dimensional unit cube, μ_j is an integer, and ϕ is real and continuous. Now, since the Koebe function corresponds to $\phi \equiv 0$, we have

$$P(a_2, \cdots, a_n) - P(2, \cdots, n) = \sum_k b_k \int f_k(t) \left[\exp\left(i \sum_{j=1}^{m_k} \mu_j \phi(t_j)\right) - 1\right] dt.$$

Hence, since b_k and f_k are real, we have

$$\begin{aligned} &|\operatorname{Re}(P(a_2, \cdots, a_n) - P(2, \cdots, n))| \\ &\leqq \text{const} \sum_k \int_{Em_k} \left[1 - \cos\left(\sum_{j=1}^{m_k} \mu_j \phi(t_j)\right)\right] dt_1 \cdots dtm_k, \end{aligned} \tag{7}$$

where Em_k denotes the m_k dimensional unit cube. By applying the Schwarz inequality, once for sums and once for integrals, we obtain

$$\begin{aligned}&|P(a_2, \cdots, a_n) - P(2, \cdots, n)|^2 \\ &\leqq \text{const} \sum_k \int (f_k(t))^2 dt \int_{E m_k} \left[1 - \cos\left(\sum_{j=1}^{m_k} \mu_j \phi(t_j)\right)\right] dt.\end{aligned} \tag{8}$$

We now note that

$$1 - \cos(x + y) \leqq 2(1 - \cos x) + 2(1 - \cos y). \tag{9}$$

This follows from

$$\begin{aligned}1 - \cos(x + y) &= 1 - \cos x + \cos x(1 - \cos y) + \sin x \sin y \\ &\leqq (1 - \cos x) + |\cos x|(1 - \cos y) + \tfrac{1}{2}(\sin^2 x + \sin^2 y).\end{aligned}$$

We next iterate (9) to obtain

$$1 - \cos\left(\sum_j \mu_j \phi(t_j)\right) \leqq \text{const} \sum_j (1 - \cos \phi(t_j)). \tag{10}$$

The proof is completed by substituting (10) into the right-hand sides of the inequalities (7) and (8) and using the fact that

$$a_2 = 2\int_0^1 e^{i\phi(t)} dt;$$

hence

$$2 - \operatorname{Re} a_2 = 2\int_0^1 (1 - \cos \phi(t)) dt.$$

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